STEADY VORTEX PATCH WITH POLYGONAL SYMMETRY FOR THE PLANAR EULER EQUATIONS IN A DISC

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Abstract. In this paper, we construct two types of vortex patch equilibria for the two-dimensional Euler equations in a disc. The first type is called the "$N+1$ type" equilibrium, in which a central vortex patch is surrounded by N identical patches with opposite signs, and the other type is called the "$2N$ type" equilibrium, in which the centers of N identical positive patches and N negative patches lie evenly on a circle. The construction is performed by solving a variational problem for the vorticity in which the kinetic energy is maximized subject to some symmetry constraints, and then analyzing the asymptotic behavior as the vorticity strength goes to infinity.

1. Introduction and main results

Let $D \subset \mathbb{R}^2$ be a simply connected bounded domain. We consider a steady ideal flow in D whose motion is governed by the following Euler equations:

$$\begin{cases} (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P & \text{in } D, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } D, \\ \mathbf{v}\cdot\mathbf{n} = 0 & \text{on } \partial D, \end{cases} \tag{1.1}$$

where $\mathbf{v} = (v_1, v_2)$ is the velocity field and P is the scalar pressure, $\mathbf{n}$ is the outward unit normal of ∂D.

Define the vorticity of the Euler flow $\omega := \partial_{x_1} v_2 - \partial_{x_2} v_1$ and take *curl* to both sides of the first equation in (1.1). Thus it becomes the following vorticity equation

$$\nabla\cdot(\omega\mathbf{v}) = 0. \tag{1.2}$$

On the other hand, since $\mathbf{v}$ is divergence free, there exists a function ψ, called the stream function, such that $\mathbf{v} = \nabla^\perp\psi := (\partial_{x_2}\psi, -\partial_{x_1}\psi)$. The boundary condition $\mathbf{v}\cdot\mathbf{n} = 0$ requires ψ to be a constant on ∂D and thus we can assume $\psi = 0$ by adding a suitable constant to ψ. Therefore by direct calculation it is not difficult to see that ω and ψ satisfy the following Poisson's equation:

$$\begin{cases} -\Delta\psi = \omega, & \text{in } D, \\ \psi = 0, & \text{on } \partial D. \end{cases} \tag{1.3}$$

So ψ can be expressed as

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$$\psi(\mathbf{x})=G\omega(\mathbf{x}):=\int_D G(\mathbf{x},\mathbf{y})\omega(\mathbf{y})d\mathbf{y}, \tag{1.4}$$

where $G(x,y)$ is the Green's function of $-\Delta$ with 0-Dirichlet boundary condition.

The energy of the flow in terms of vorticity ω is defined as

$$E(\omega):=\frac{1}{2}\int_D\int_D G(\mathbf{x},\mathbf{y})\omega(\mathbf{x})\omega(\mathbf{y})d\mathbf{x}d\mathbf{y}. \tag{1.5}$$

The research for dynamically possible equilibria of planar incompressible flows and their stability analysis have been extensively studied by many authors; see [1, 2, 3, 5, 8, 9, 10, 12, 13, 14, 17, 27, 22, 24, 26, 28, 29, 30, 31] and the references therein. It is also worth mentioning that in [32] the authors studied $N+1$ type vortex patch equilibrium numerically based on contour dynamics, and stability is also discussed therein.

Since in this paper, we are going to deal with vortex patch solution, namely, the vorticity of the fluid is a piecewise constant function, it is necessary to interpret (1.6) in a weak sense. To do this we first reformulate the Euler equations (1.1) in terms of the vorticity as follows

$$\nabla\cdot(\omega\nabla^{\perp}G\omega)=0 \quad \text{in } D. \tag{1.6}$$

We will mainly consider (1.6) instead of (1.1) in the rest of this paper.

Definition 1.1. We call $\omega\in L^{\frac{4}{3}}(D)$ a weak solution to (1.6) if

$$\int_D \omega\nabla^{\perp}G\omega\cdot\nabla\phi d\mathbf{x}=0 \tag{1.7}$$

for any $\phi\in C_0^\infty(D)$.

To state our main results we need to introduce some definitions first. It is well known that G has the following decomposition

$$G(\mathbf{x},\mathbf{y}):=\frac{1}{2\pi}\ln\frac{1}{|\mathbf{x}-\mathbf{y}|}-h(\mathbf{x},\mathbf{y}),\ \ \mathbf{x},\mathbf{y}\in D, \tag{1.8}$$

where $h(\mathbf{x},\mathbf{y})$ is the regular part of G. We will call $h(\mathbf{x},\mathbf{x})$ Robin function of D.

Let k be a positive integer and $\kappa_1,\kappa_2,...,\kappa_k$ be k non-zero real numbers. The corresponding Kirchhoff-Routh function H_k is defined by setting:

$$H_k(\mathbf{x_1},\cdots,\mathbf{x_k}):=-\sum_{1\le i\neq j\le k}\kappa_i\kappa_j G(\mathbf{x_i},\mathbf{x_j})+\sum_{i=1}^{k}\kappa_i^2 h(\mathbf{x_i},\mathbf{x_i}), \tag{1.9}$$

where $(\mathbf{x}_1,\cdots,\mathbf{x}_k)\in D^{(k)}:=\underbrace{D\times D\times\cdots\times D}_{k}$ such that $\mathbf{x}_i\neq\mathbf{x}_j$ for $i\neq j$.

In [28, 29], Turkington firstly proved existence of steady vortex patches in general bounded domains. He introduced a weakly star closed subset in L^∞ that contains all "isovortical" patches, and by maximizing the kinetic energy on that subset he showed that any maximizer is in fact a steady vortex patch. He also considered asymptotic behavior of the maximizers as the vorticity strength goes to infinity and found that the limit is a point vortex located at a global minimum point of the Robin function of the domain. Later Burton set up a complete theory on the extremization of the energy functional on "isovortical" rearrangements; see [7, 8, 9, 10]. In particular, he proved existence of maximizer, minimizer and minimax solutions of the corresponding variational problems, including Turkington's result as a special case. But Burton did not consider the case when the vorticity strength goes the infinity, which has practical interest and is closely related to the desingularization of point vortices; see [30]. As an application of Burton's theory and by using Turkington's asymptotic estimate, Elcrat and Miller [18, 19] considered that case and constructed steady multiple vortex flows near any given strict local minimum point of the Kirchhoff-Routh function.

A crucial assumption in [18, 19] is that the minimum point of the Kirchhoff-Routh function must be strict, which is usually difficult to verify. In fact, there is no general result that guarantees the existence of such minimum point except for several special cases; see Remark 3.1 in [15]. For the simplest domain, an open disc, due to rotational invariance of the Kirchhoff-Routh function, there is no strict local minimum point for more than two vortices. To circumvent this difficulty, the authors in [16] improved the vorticity method by adding some symmetry constraints in the variational problem. They constructed steady vortex patches, each of which consists of a positive part and a negative part, concentrating near two antipodal points respectively. Moreover, the combination of these two points is the unique minimum point of the corresponding Kirchhoff-Routh function up to rotational invariance.

Our aim in this paper is to extend the result in [16] to the case with multiple patches concentrating near some prescribed points. To state our main results, we need to introduce more notations and terminology which will also be used later. From now on D will be the unit disc centered at the origin in $\mathbb{R}^2$, that is

$$D = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \mid |\mathbf{x}| = \sqrt{x_1^2 + x_2^2} < 1\}, \tag{1.10}$$

and the regular part of G in this case is

$$h(\mathbf{x}, \mathbf{y}) := -\frac{1}{2\pi} \ln |\mathbf{y}| - \frac{1}{2\pi} \ln |\mathbf{x} - \frac{\mathbf{y}}{|\mathbf{y}|^2}|, \ \ \mathbf{x}, \mathbf{y} \in D. \tag{1.11}$$

We shall use the following notation for convenience. For any measurable set $A \subset D$, I_A denotes the characteristic function of A, that is, $I_A = 1$ on A and $I_A = 0$ elsewhere, $|A|$ denotes the two-dimensional Lebesgue measure of A, and $\overline{A}$ denotes the closure of A in the usual Euclidean topology.

Throughout the sequel we shall use the following notations. Let N be a positive integer. For any $\mathbf{x} \in D$ with $\mathbf{x} = \rho_{\mathbf{x}}(\cos\theta_{\mathbf{x}}, \sin\theta_{\mathbf{x}})$, where $[\rho_{\mathbf{x}}, \theta_{\mathbf{x}}]$ denotes the polar coordinates of $\mathbf{x}$, define

$$\mathbf{e}^{\mathbf{i}\theta}(\mathbf{x}) := \rho_{\mathbf{x}}(\cos(\theta_{\mathbf{x}} + \theta), \sin(\theta_{\mathbf{x}} + \theta)),$$

$$\mathcal{R}_i(\mathbf{x}) := \rho_{\mathbf{x}}(\cos(\frac{4\pi(i-1)}{N} - \theta_{\mathbf{x}}), \sin(\frac{4\pi(i-1)}{N} - \theta_{\mathbf{x}})), \ \ i = 1, \cdots, N,$$

$$\mathcal{T}_j(\mathbf{x}) := \rho_{\mathbf{x}}(\cos(\frac{2\pi(j-1)}{N} - \theta_{\mathbf{x}}), \sin(\frac{2\pi(j-1)}{N} - \theta_{\mathbf{x}})),\ j = 1, \cdots, 2N.$$

It is easy to see that $\mathbf{e}^{\mathbf{i}\theta}(\mathbf{x})$ represents the anticlockwise rotation through θ of $\mathbf{x}$, $\mathcal{R}_i(\mathbf{x})$ is the reflection point of $\mathbf{x}$ with respect to the line $x_2 = \big(\tan\frac{2\pi(i-1)}{N}\big)x_1$, and $\mathcal{T}_j(\mathbf{x})$ is the reflection point of $\mathbf{x}$ with respect to the line $x_2 = \big(\tan\frac{\pi(j-1)}{N}\big)x_1$.

Let f be a real-valued function defined in D, define $\mathbf{e}^{\mathbf{i}\theta}(f)(\mathbf{x}) := f(\mathbf{e}^{\mathbf{i}\theta}(\mathbf{x}))$, $\mathcal{R}_i(f)(\mathbf{x}) := f(\mathcal{R}_i(\mathbf{x}))$ for each $i = 1, \cdots, N$ and $\mathcal{T}_j(f)(\mathbf{x}) := f(\mathcal{T}_j(\mathbf{x}))$ for each $j = 1, \cdots, 2N$. Let A be a subset in D, define $\mathbf{e}^{\mathbf{i}\theta}(A) := \{\mathbf{e}^{\mathbf{i}\theta}(\mathbf{x}) \mid \mathbf{x} \in A\}$, $\mathcal{R}_i(A) := \{\mathcal{R}_i(\mathbf{x}) \mid \mathbf{x} \in A\}$ and $\mathcal{T}_j(A) := \{\mathcal{T}_j(\mathbf{x}) \mid \mathbf{x} \in A\}$.

Now we are ready to state the main results of this paper. The first one is the following.

Theorem 1.2. *Let N be a positive integer, κ, μ be two positive real numbers such that μ/κ is sufficiently large(depending only on N). Then there exists a $\lambda_0 > 0$, such that for any $\lambda > \lambda_0$, there exists a weak solution ω^λ of* (1.6) *with the following properties:*

(1) $\omega^\lambda = \lambda I_{\Omega_0^\lambda} - \sum_{i=1}^N \lambda I_{\Omega_i^\lambda}$ *with* $\Omega_0^\lambda = \big\{x \in D \mid G\omega^\lambda(x) > \nu_0^\lambda\big\} \cap B_{\delta_0}(Q_{N+1})$ *and* $\Omega_i^\lambda = \big\{x \in D \mid G\omega^\lambda(x) < -\nu_1^\lambda\big\} \cap B_{\delta_0}(Q_i)$, *where* $\nu_0^\lambda, \nu_1^\lambda$ *are positive numbers depending on* λ, $Q_i = \tilde{\rho}\left(\cos\frac{2\pi(i-1)}{N}, \sin\frac{2\pi(i-1)}{N}\right)$ *for some* $\tilde{\rho} \in (0,1)$ *determined only by* N *and* κ/μ, $Q_{N+1} = \mathbf{0}$ *and* δ_0 *is chosen to be sufficiently small such that* $\overline{B_{\delta_0}(Q_i)} \cap \overline{B_{\delta_0}(Q_j)} = \varnothing$ *if* $1 \le i \ne j \le N+1$;

(2) $\lambda|\Omega_0^\lambda| = \mu, \lambda|\Omega_i^\lambda| = \kappa$ *for* $i = 1, \cdots, N$;

(3) $\mathbf{e}^{\mathbf{i}\frac{2\pi(i-1)}{N}}(\omega^\lambda) = \omega^\lambda$ *and* $\mathcal{R}_i(\omega^\lambda) = \omega^\lambda$ *for* $i = 1, \cdots, N$;

(4) Ω_0^λ *shrinks to the origin* $\mathbf{0}$ *and* Ω_i^λ *shrinks to* Q_i *for each* $i = 1, \cdots, N$ *as* $\lambda \to +\infty$, *or equivalently,*

$$\lim_{\lambda\to+\infty}\sup_{\mathbf{x}\in\Omega_0^\lambda}|\mathbf{x}| = 0,\ \lim_{\lambda\to+\infty}\sup_{\mathbf{x}\in\Omega_i^\lambda}|\mathbf{x} - Q_i| = 0,\ i = 1, \cdots, N.$$

Let us remark that Theorem 1.2 deals with $N+1$ type vortex patch equilibrium, in which a central positive patch is surrounded by N identical negative patches. By the physical meaning of ω, Theorem 1.2 implies the existence of flow with one piece of fluid rotating at the angular velocity of $\frac{1}{2}\lambda$ and another N pieces of fluid rotating at the angular velocity of $-\frac{1}{2}\lambda$ which almost evenly distributed on the circle centered at the origin with radius determined only by N for λ large.

The second result is the following theorem which is concerned with the existence of $2N$ type vortex patch equilibrium in D. More precisely, we construct steady vortex patches, each of which consists of N positive patches and N negative patches whose centers lie evenly near $2N$ different points on a circle centered at the origin radius determined only by N. As before one can get the physical interpretation.

Theorem 1.3. *Let N be a positive integer. Then there exists $\lambda_0 > 0$, such that for any $\lambda > \lambda_0$, there exists a weak solution ω^λ of* (1.6) *with the following properties:*

(1) $\omega^\lambda = \sum_{i=1}^{2N}(-1)^{i-1}\lambda I_{U_i^\lambda}$ *with* $U_i^\lambda = \big\{x \in D \mid (-1)^{i-1}G\omega^\lambda(x) > \tau^\lambda\big\} \cap B_{\delta_0}(P_i)$ *for* $i = 1, \cdots, 2N$, *where* $\tau^\lambda \in \mathbb{R}^+$ *depends on* λ, $P_i = \bar{\rho}\left(\cos\frac{\pi(i-1)}{N}, \sin\frac{\pi(i-1)}{N}\right)$ *for some* $\bar{\rho} \in$

$(0,1)$ *determined by* N*, and* δ_0 *is chosen to be sufficiently small such that* $\overline{B_{\delta_0}(P_i)}\cap\overline{B_{\delta_0}(P_j)}=\varnothing$ *if* $i\neq j$*;*

(2) $\lambda|U_i^\lambda|=1$ *for* $i=1,\cdots,2N$*;*

(3) $\mathbf{e}^{\mathbf{i}\frac{\pi(i-1)}{N}}(\omega^\lambda)=(-1)^{i-1}\omega^\lambda$ *and* $\mathcal{T}_i(\omega^\lambda)=\omega^\lambda$ *for* $i=1,\cdots,2N$*;*

(4) *for each* $1\leq i\leq 2N$*,* U_i^λ *shrinks to* P_i *as* $\lambda\to+\infty$*, or equivalently,*

$$\lim_{\lambda\to+\infty}\sup_{\mathbf{x}\in U_i^\lambda}|\mathbf{x}-P_i|=0.$$

The outline of proofs for the two results is as follows. First we calculate the minimum point of the Kirchhoff-Routh function subject to some symmetry constraints. Then we solve a variational problem for the vorticity in which some symmetry constraints are added in the admissible class originally used by Elcrat and Miller in [19]. Finally by analyzing the asymptotic behavior of the maximizers we show that they are in fact steady solutions to the Euler equation.

This paper is organized as follows. In Section 2 we give proof of Theorem 1.2. In Section 3 we give a sketch of the proof for Theorem 1.3.

2. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2.

Throughout this section we choose $k=N+1$ in (1.9), where N is a given positive integer. We also assume that $\kappa_i=-\kappa$ for $i=1,\cdots,N$ and $\kappa_{N+1}=\mu$ for two given positive real numbers κ,μ. From (1.9), the Kirchhoff-Routh function is

$$\begin{aligned}H_{N+1}(\mathbf{x}_1,\cdots,\mathbf{x}_{N+1})=&-\kappa^2\sum_{i\neq j,1\leq i,j\leq N}G(\mathbf{x}_i,\mathbf{x}_j)+2\kappa\mu\sum_{i=1}^{N}G(\mathbf{x}_i,\mathbf{x}_{N+1})\\&+\sum_{i=1}^{N}\kappa^2h(\mathbf{x}_i,\mathbf{x}_i)+\mu^2h(\mathbf{x}_{N+1},\mathbf{x}_{N+1}).\end{aligned}\tag{2.1}$$

For the discussion in the sequel, define

$$\begin{aligned}\mathcal{S}_{N+1}:=\Big\{&(\mathbf{x}_1,\cdots,\mathbf{x}_{N+1})\in D^{(N+1)}\mid \mathbf{x}_{N+1}=\mathbf{0},\\&\mathbf{x}_i=\rho\left(\cos\frac{2\pi(i-1)}{N},\sin\frac{2\pi(i-1)}{N}\right),1\leq i\leq N,\rho\in(0,1)\Big\}.\end{aligned}\tag{2.2}$$

2.1. Constrained minimum point of the Kirchhoff-Routh function.

First we show that H_{N+1} has a unique minimum point on $\mathcal{S}_{N+1}$, which depends only on N, provided that μ/κ is sufficiently large.

Lemma 2.1. *There exists* $C_0>0$*, depending only on* N*, such that for any* $\mu,\kappa>0$ *with* $\mu/\kappa>C_0$*, there exists a unique minimum point for* H_{N+1} *on* $\mathcal{S}_{N+1}$*.*

Proof. On $\mathcal{S}_{N+1}$, H_{N+1} is the function of ρ for $\rho \in (0,1)$. To prove the lemma, it suffices to show that $\lim_{\rho\to 0^+} H_{N+1} = +\infty, \lim_{\rho \to 1^-} H_{N+1} = +\infty$ and H_{N+1} is strictly convex in $(0,1)$ if μ/κ is sufficiently large.

By direct calculation, we get

$$
\begin{aligned}
&H_{N+1}(\mathbf{x}_1,\cdots,\mathbf{x}_{N+1})\\
&= -\sum_{1\le i\neq j\le N,}\kappa^2 G(\mathbf{x}_i,\mathbf{x}_j) + 2\sum_{i=1}^{N}\kappa\mu G(\mathbf{x}_i,\mathbf{x}_{N+1}) + \sum_{i=1}^{N}\kappa^2 h(\mathbf{x}_i,\mathbf{x}_i) + \mu^2 h(\mathbf{x}_{N+1},\mathbf{x}_{N+1})\\
&\qquad = 2\kappa\mu\sum_{i=1}^{N}\left[-\frac{1}{2\pi}\ln|\mathbf{x}_i-\mathbf{x}_{N+1}| + \frac{1}{2\pi}\ln|\mathbf{x}_i| + \frac{1}{2\pi}\ln|\mathbf{x}_{N+1}-\frac{\mathbf{x}_i}{|\mathbf{x}_i|^2}|\right]\\
&\qquad - \kappa^2\sum_{1\le i\neq j\le N}\left[-\frac{1}{2\pi}\ln|\mathbf{x}_i-\mathbf{x}_j| + \frac{1}{2\pi}\ln|\mathbf{x}_j| + \frac{1}{2\pi}\ln|\mathbf{x}_i-\frac{\mathbf{x}_j}{|\mathbf{x}_j|^2}|\right]\\
&\qquad + \kappa^2\sum_{i=1}^{N}h(\mathbf{x}_i,\mathbf{x}_i) + \mu^2 h(\mathbf{x}_{N+1},\mathbf{x}_{N+1})\\
&\qquad := A_1 + A_2 + A_3.
\end{aligned}
$$

For simplicity we write $\mathbf{x}_i = \rho\left(\cos\frac{2\pi(i-1)}{N}, \sin\frac{2\pi(i-1)}{N}\right)$ for $i=1,\cdots,N$, then we get

$$
\begin{aligned}
&|\mathbf{x}_i-\mathbf{x}_j| = 2\rho^2\left(1-\cos\frac{2\pi(i-j)}{N}\right), \qquad \left|\mathbf{x}_i - \frac{\mathbf{x}_i}{|\mathbf{x}_i|^2}\right| = \rho - \frac{1}{\rho},\\
&\left|\mathbf{x}_i - \frac{\mathbf{x}_j}{|\mathbf{x}_j|^2}\right| = \rho^2 + \frac{1}{\rho^2} - 2\cos\frac{2\pi(i-j)}{N}, \quad h(\mathbf{0},\mathbf{0}) = 0, \quad h(\mathbf{x},\mathbf{x}) = -\frac{1}{2\pi}\ln(1-|\mathbf{x}|^2).
\end{aligned}
$$

It is easy to check that

$$
\begin{aligned}
A_1 &= \frac{N\kappa\mu}{\pi}\ln\frac{1}{\rho},\\
A_2 &= -\kappa^2\sum_{1\le i\neq j\le N}\Bigg[-\frac{1}{4\pi}\ln\left(2\rho^2(1-\cos\frac{2\pi(i-j)}{N})\right) + \frac{1}{2\pi}\ln\rho\\
&\quad + \frac{1}{4\pi}\ln\left(\rho^2+\frac{1}{\rho^2}-2\cos(\frac{2\pi(i-j)}{N})\right)\Bigg],\\
A_3 &= -\frac{N\kappa^2}{2\pi}\ln\left(1-\rho^2\right),
\end{aligned}
$$

from which we obtain

$$
\begin{aligned}
H_{N+1}(\rho) = \frac{N\kappa\mu}{\pi}\ln\frac{1}{\rho} - \kappa^2\sum_{1\le i\neq j\le N}\Bigg[-\frac{1}{4\pi}\ln\left(2\rho^2(1-\cos\frac{2\pi(i-j)}{N})\right)\\
+\frac{1}{2\pi}\ln\rho + \frac{1}{4\pi}\ln\left(\rho^2+\frac{1}{\rho^2}-2\cos(\frac{2\pi(i-j)}{N})\right)\Bigg] - \frac{N\kappa^2}{2\pi}\ln(1-\rho^2).
\end{aligned}
\tag{2.3}
$$

Denote $\gamma := \mu/\kappa$, then

$$
\begin{aligned}
H_{N+1}(\rho) = \frac{\kappa^2}{2\pi}\Big[&- 2N\gamma \ln\rho - \sum_{1\le i\neq j\le N} \big[-\ln\rho - \frac{1}{2}\ln\big(2-2\cos\frac{2\pi(i-j)}{N}\big) + \ln\rho \\
&+ \frac{1}{2}\ln\big(\rho^2+\frac{1}{\rho^2}-2\cos\frac{2\pi(i-j)}{N}\big)\big] - N\ln(1-\rho^2)\Big] \\
= \frac{\kappa^2}{2\pi}\Big[&- 2N\gamma\ln\rho - N\ln(1-\rho^2) - \sum_{1\le i\neq j\le N}\big[-\frac{1}{2}\ln\big(2-2\cos\frac{2\pi(i-j)}{N}\big) \\
&+ \frac{1}{2}\ln\big(\rho^2+\frac{1}{\rho^2}-2\cos\frac{2\pi(i-j)}{N}\big)\big]\Big] \\
:= \frac{\kappa^2}{2\pi}&f(\rho),
\end{aligned}
$$

where

$$
\begin{aligned}
f(\rho) &= -2N\gamma\ln\rho - N\ln(1-\rho^2) - \sum_{1\le i\neq j\le N}\left[-\frac{1}{2}\ln\left(2-2\cos\frac{2\pi(i-j)}{N}\right)\right. \\
&\quad \left. + \frac{1}{2}\ln\left(\rho^2+\frac{1}{\rho^2}-2\cos\frac{2\pi(i-j)}{N}\right)\right] \\
&= -N\ln\big(\rho^{2\gamma}(1-\rho^2)\big) - \frac{1}{2}\sum_{1\le i\neq j\le N}\ln\left(\rho^2+\frac{1}{\rho^2}-2\cos\frac{2\pi(i-j)}{N}\right) \\
&\quad + \frac{1}{2}\sum_{1\le i\neq j\le N}\ln\left(2-2\cos\frac{2\pi(i-j)}{N}\right).
\end{aligned}
$$

Using the above expression of $f(\rho)$, it is easy to prove the following properties:

(1) $\lim_{\rho\to 1^-} f(\rho) = +\infty$, $\lim_{\rho\to 0^+} f(\rho) = +\infty$;
(2) $f''(\rho) > 0$ for $\rho\in(0,1)$, provided γ sufficiently large.

Since the proof is elementary we will not go into details here. Indeed, as parameter γ sufficiently large, one shows that the function f/γ is a small C^2 perturbation of a strictly convex coercive function $-2N\ln\rho$, and hence is itself convex and coercive.

Thus, there must be a unique global minimum point $\tilde\rho\in(0,1)$ of $H_{N+1}(\rho)$ on $\mathcal{S}_{N+1}$, which completes the proof of Lemma 2.1.

□

2.2. **Variational problem.** From now on we assume that μ/κ is sufficiently large such that there is a unique minimum point of H_{N+1} on $\mathcal{S}_{N+1}$, say $(Q_1,\cdots,Q_N,Q_{N+1})$, with $Q_i = \tilde\rho\left(\cos\frac{2\pi(i-1)}{N}, \sin\frac{2\pi(i-1)}{N}\right)$ for $i=1,\cdots,N$ and $Q_{N+1}=\mathbf{0}$. $\tilde\rho$ is uniquely determined by N and μ/κ.

We also choose $\delta_0>0$ sufficiently small such that

$$\begin{cases} B_i:=B_{\delta_0}(Q_i)\subset\subset D, i=1,\cdots,N+1,\\ \overline{B_{\delta_0}(Q_i)}\cap\overline{B_{\delta_0}(Q_j)}=\varnothing, 1\leq i\neq j\leq N+1,\\ B_{\delta_0}(Q_i)\subset\subset\{(\rho,\theta)\mid \frac{2\pi(i-1)}{N}-\frac{\pi}{N}<\theta<\frac{2\pi(i-1)}{N}-\frac{\pi}{N}\}, i=1,\cdots,N.\end{cases} \tag{2.4}$$

Here $[\rho,\theta]$ represents some point in polar coordinates.

Define the following admissible class:

$$\begin{aligned} M^\lambda(D):=\Big\{&\omega\in L^\infty(D)\mid \omega=\sum_{i=1}^{N+1}\omega_i, supp\omega_i\subset B_i, 0\leq\omega_{N+1}\leq\lambda, -\lambda\leq\omega_i\leq 0,\\ &\int_D\omega_i(\mathbf{x})d\mathbf{x}=-\kappa \text{ for } i=1,\cdots,N,\ \int_D\omega_{N+1}(\mathbf{x})d\mathbf{x}=\mu,\\ &\mathbf{e}^{\mathbf{i}\frac{2\pi(i-1)}{N}}(\omega)=\omega \text{ and } \mathcal{R}_i(\omega)=\omega \text{ for } i=1,\cdots,N\Big\}.\end{aligned}$$

Remark 2.2. It is easy to see that $M^\lambda(D)$ is not empty when λ is sufficiently large. In fact,

$$\lambda I_{B_{\varepsilon_2}(\mathbf{0})}-\lambda\sum_{i=1}^N I_{B_{\varepsilon_1}(Q_i)}\in M^\lambda(D),$$

where $\lambda\pi\varepsilon_1^2=\kappa, \lambda\pi\varepsilon_2^2=\mu$. By the symmetry of ω and Green's function G, we also note that for any $\omega\in M^\lambda(D)$, the stream function $\psi:=G\omega$ satisfies

$$\mathbf{e}^{\mathbf{i}\frac{2\pi(i-1)}{N}}(\psi)=\psi,\ \mathcal{R}_i(\psi)=\psi.$$

Let $E(\omega)$, the energy of the vorticity ω, be as defined in (1.5). First, we show the existence of maximizers of $E(\omega)$ on the admissible class $M^\lambda(D)$.

Lemma 2.3. *There exists $\omega^\lambda\in M^\lambda(D)$, such that $E(\omega^\lambda)=\sup_{\omega\in M^\lambda(D)}E(\omega)$.*

Proof. Since $G(\mathbf{x},\mathbf{y})\in L^1(D\times D)$, we have

$$E(\omega)=\frac{1}{2}\int_D\int_D G(\mathbf{x},\mathbf{y})\omega(\mathbf{x})\omega(\mathbf{y})d\mathbf{x}d\mathbf{y}\leq\frac{\lambda^2}{2}\int_D\int_D|G(\mathbf{x},\mathbf{y})|d\mathbf{x}d\mathbf{y}<+\infty, \tag{2.5}$$

which means that $E(\omega)$ is bounded from above.

Now let $\{\omega_n\}$ be a maximizing sequence of E, that is, $\lim_{n\to\infty}E(\omega_n)=\sup_{\omega\in M^\lambda(D)}E(\omega)$. Since $|\omega_n|\leq\lambda$ and the unit disc of $L^\infty(D)$ is sequentially compact in weakly star topology in $L^\infty(D)$, there exists $\omega^\lambda\in L^\infty(D)$ such that $\omega_n\to\omega^\lambda$ weakly star in $L^\infty(D)$(up to a subsequence). Now we show that $\omega^\lambda\in M^\lambda(D)$. In fact, by the definition of convergence in weakly star topology,

$$\lim_{n\to\infty}\int_D\omega_n(\mathbf{x})\phi(\mathbf{x})d\mathbf{x}=\int_D\omega^\lambda(\mathbf{x})\phi(\mathbf{x})d\mathbf{x},\qquad \text{for any } \phi\in L^1(D). \tag{2.6}$$

(1) By taking $\phi \in L^1(D)$ such that $supp\phi \subset D \setminus \cup_{i=1}^{N+1} B_{\delta_0}(Q_i)$ in (2.6), we get

$$\int_D \omega^\lambda(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = \lim_{n\to\infty}\int_D \omega_n(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = 0, \tag{2.7}$$

which implies

$$supp\omega^\lambda \subset \bigcup_{i=1}^{N+1} B_{\delta_0}(Q_i). \tag{2.8}$$

(2) Define $\omega_i^\lambda = \omega^\lambda I_{B_{\delta_0}(Q_i)}$. Taking $\phi = I_{B_{\delta_0}(Q_i)} \in L^1(D)$ in (2.6), we get

$$\int_D \omega_i^\lambda(\mathbf{x})d\mathbf{x} = \int_D \omega^\lambda(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = \lim_{n\to\infty}\int_D \omega_n(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = -\kappa,$$

for $i = 1, \cdots, N$ and

$$\int_D \omega_{N+1}^\lambda(\mathbf{x})d\mathbf{x} = \int_D \omega^\lambda(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = \lim_{n\to\infty}\int_D \omega_n(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = \mu.$$

(3) Now we prove $0 \le \omega_{N+1}^\lambda \le \lambda$ and $-\lambda \le \omega_i^\lambda \le 0$ a.e. in D for $i = 1, \cdots, N$. First, we prove $\omega_{N+1}^\lambda \le \lambda$ a.e. in D. Assume that $\left|\left\{\mathbf{x} \in D \mid \omega_{N+1}^\lambda(\mathbf{x}) > \lambda\right\}\right| > 0$, then there exists $\delta_1 > 0$ and $\varepsilon_0 > 0$ such that $\left|\left\{\mathbf{x} \mid \omega_{N+1}^\lambda(\mathbf{x}) - \lambda > \delta_1\right\}\right| > \varepsilon_0$. Take $\phi = I_{\left\{\mathbf{x}\in D \mid \omega_{N+1}^\lambda(\mathbf{x})-\lambda>\delta_1\right\}} \in L^1(D)$ into (2.6), then we get

$$\begin{aligned} 0 \ge &\lim_{n\to\infty}\int_D (\omega_n(\mathbf{x}) - \lambda)\phi(\mathbf{x})d\mathbf{x} = \int_D \left(\omega^\lambda(\mathbf{x}) - \lambda\right)\phi(\mathbf{x})d\mathbf{x} \\ >&\delta_1\left|\left\{\mathbf{x} \mid \omega_{N+1}^\lambda(\mathbf{x}) - \lambda > \delta_1\right\}\right| > \delta_1\varepsilon_0, \end{aligned} \tag{2.9}$$

which is an obvious contradiction. Thus we get $\omega_{N+1}^\lambda \le \lambda$ a.e. in D. Similarly, we can prove $\omega_{N+1}^\lambda \ge 0$ a.e. in D. Other conclusions can be proved similarly.

(4) We prove $\mathbf{e}^{\mathbf{i}\frac{2\pi(i-1)}{N}}\left(\omega^\lambda\right) = \omega^\lambda$ for $i = 1, \cdots, N$. For any $\mathbf{x_0} \in D$, by taking $\phi = \frac{1}{\pi s^2}\left[I_{B_s(\mathbf{x_0})} - I_{B_s\left(\mathbf{e}^{\mathbf{i}\frac{2\pi(i-1)}{N}}(\mathbf{x_0})\right)}\right]$ in (2.6) for small $s > 0$, we get

$$\begin{aligned} 0 =& \lim_{n\to\infty}\int_D \omega_n(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = \int_D \omega^\lambda(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} \\ =&\frac{1}{\pi s^2}\int_{B_s(\mathbf{x_0})}\omega^\lambda(\mathbf{x})\,d\mathbf{x} - \frac{1}{\pi s^2}\int_{B_s\left(\mathbf{e}^{\mathbf{i}\frac{2\pi(i-1)}{N}}(\mathbf{x_0})\right)}\omega^\lambda(\mathbf{x})\,d\mathbf{x}, \end{aligned}$$

that is,

$$\frac{1}{\pi s^2}\int_{B_s(\mathbf{x_0})}\omega^\lambda(\mathbf{x})\,d\mathbf{x} = \frac{1}{\pi s^2}\int_{B_s\left(\mathbf{e}^{\mathbf{i}\frac{2\pi(i-1)}{N}}(\mathbf{x_0})\right)}\omega^\lambda(\mathbf{x})\,d\mathbf{x}.$$

By Lebesgue's differential theorem, we get

$$\mathbf{e}^{\mathbf{i}\frac{2\pi(i-1)}{N}}\left(\omega^\lambda\right)(\mathbf{x_0}) = \omega^\lambda(\mathbf{x_0}) \qquad \text{a.e in} \;\; D. \tag{2.10}$$

Similarly, we can also prove $\mathcal{R}_i\left(\omega^\lambda\right)=\omega^\lambda$ a.e. in D for $i=1,\cdots,N$.

Therefore, we get $\omega^\lambda\in M^\lambda(D)$. To finish the proof, it suffices to note that E is weakly star continuous on $M^\lambda(D)$. In fact, since $\omega_n\to\omega^\lambda$ weakly star in $L^\infty(D)$, we deduce that $\omega_n\to\omega^\lambda$ in weak topology in $L^p(D)$ for any $p>1$, so we get

$$G\omega_n\to G\omega^\lambda \text{ weakly in } W^{2,p}(D). \tag{2.11}$$

By Rellich's compact embedding theorem, we get $G\omega_n\to G\omega^\lambda$ in $L^{p'}(D)$, where $p'=\frac{p}{p-1}$. Therefore

$$\lim_{n\to\infty}E(\omega_n)=\lim_{n\to\infty}\int_D\omega_n(\mathbf{x})G\omega_n(\mathbf{x})d\mathbf{x}=\int_D\omega^\lambda(\mathbf{x})\,G\omega^\lambda(\mathbf{x})\,d\mathbf{x}=E\left(\omega^\lambda\right), \tag{2.12}$$

which completes the proof.

□

Remark 2.4. From now on, we use $\psi^\lambda:=G\omega^\lambda$ to denote the stream function of ω^λ. Moreover, since $W^{2,p}(D)$ is continuously embedded into $C^\alpha(\overline{D})$ for $\alpha\in(0,1)$, we know that $\psi^\lambda\in C^\alpha(\overline{D})$.

2.3. **Profile of ω^λ.** In this subsection, we show that the maximizer ω^λ is in fact a vortex patch and has a explicit form.

Lemma 2.5. *Let ω^λ be a maximizer of $E(\omega)$ on $M^\lambda(D)$, then there exist $\nu_0^\lambda,\nu_1^\lambda\in\mathbb{R}^+$ depending on λ, such that*

$$\omega^\lambda=\lambda I_{\left\{\mathbf{x}\in D|\psi^\lambda(\mathbf{x})>\nu_0^\lambda\right\}\cap B_{N+1}}-\lambda\sum_{i=1}^N I_{\left\{\mathbf{x}\in D|\psi^\lambda(\mathbf{x})<-\nu_1^\lambda\right\}\cap B_i}. \tag{2.13}$$

Proof. Define $\omega_i^\lambda=\omega^\lambda I_{B_i}$, then obviously $\omega^\lambda=\sum_{i=1}^{N+1}\omega_i^\lambda$. It suffices to show that $\omega_{N+1}^\lambda=\lambda I_{\{\mathbf{x}\in D|\psi^\lambda(\mathbf{x})>\nu_0^\lambda\}\cap B_{N+1}}$ for some $\nu_0^\lambda\in\mathbb{R}^+$, since we can use similar methods to calculate ω_i^λ.

For any $z_1,z_2\in L^\infty(D)$ satisfying

$$\begin{cases} suppz_1, suppz_2\subset B_{N+1},\\ z_1,z_2\geq 0, \text{ a.e. in } D, \int_D z_1(\mathbf{x})d\mathbf{x}=\int_D z_2(\mathbf{x})d\mathbf{x},\\ \mathcal{R}_i(z_1)=z_1,\ \mathcal{R}_i(z_2)=z_2 \text{ for } i=1,\cdots,N,\\ \mathbf{e}^{\mathbf{i}\frac{2\pi(i-1)}{N}}(z_1)=z_1,\ \mathbf{e}^{\mathbf{i}\frac{2\pi(i-1)}{N}}(z_2)=z_2 \text{ for } i=1,\cdots,N,\\ z_1=0 \quad\text{in}\quad D\setminus\left\{\mathbf{x}\in D\mid\omega^\lambda(\mathbf{x})\leq\lambda-a\right\},\\ z_2=0 \quad\text{in}\quad D\setminus\left\{\mathbf{x}\in D\mid\omega^\lambda(\mathbf{x})\geq a\right\}. \end{cases} \tag{2.14}$$

where $a>0$ is sufficiently small, we define a family of test functions $\omega_s=\omega^\lambda+s(z_1-z_2)$, $s>0$. It is easy to check that if s is sufficiently small(depending on $a,||z_1||_{L^\infty(D)}$ and $||z_2||_{L^\infty(D)}$), then $\omega_s\in M^\lambda(D)$. So we have

$$\frac{dE(\omega_s)}{ds}|_{s=0^+}\leq 0. \tag{2.15}$$

Repeating the previous argument we get

$$\int_D \psi^\lambda(\mathbf{x})\, z_1(\mathbf{x}) d\mathbf{x} \le \int_D \psi^\lambda(\mathbf{x})\, z_2(\mathbf{x}) d\mathbf{x},$$

from which we deduce that

$$\sup_{\left\{\mathbf{x}\in D|\omega^\lambda(\mathbf{x})<\lambda\right\}\cap B_{N+1}\cap\left\{(\rho,\theta)|0<\theta<\frac{\pi}{N}\right\}} \psi^\lambda(\mathbf{x}) = \inf_{\left\{\mathbf{x}\in D|\omega^\lambda(\mathbf{x})>0\right\}\cap B_{N+1}\cap\left\{(\rho,\theta)|0<\theta<\frac{\pi}{N}\right\}} \psi^\lambda(\mathbf{x}).$$

Therefore, if we define

$$\nu_0^\lambda := \sup_{\left\{\mathbf{x}\in D|\omega^\lambda(\mathbf{x})<\lambda\right\}\cap B_{N+1}\cap\left\{(\rho,\theta)|0<\theta<\frac{\pi}{N}\right\}} \psi^\lambda(\mathbf{x}) = \inf_{\left\{\mathbf{x}\in D|\omega^\lambda(\mathbf{x})>0\right\}\cap B_{N+1}\cap\left\{(\rho,\theta)|0<\theta<\frac{\pi}{N}\right\}} \psi^\lambda(\mathbf{x}),$$

it is easy to check that

$$\begin{cases} \omega^\lambda = \lambda & \text{a.e. in} \quad B_{N+1}\cap\left\{(\rho,\theta)\mid 0<\theta<\frac{\pi}{N}\right\}\cap\left\{\mathbf{x}\in D\mid \psi^\lambda(\mathbf{x})>\nu_0^\lambda\right\}, \\ \omega^\lambda = 0 & \text{a.e. in} \quad B_{N+1}\cap\left\{(\rho,\theta)\mid 0<\theta<\frac{\pi}{N}\right\}\cap\left\{\mathbf{x}\in D\mid \psi^\lambda(\mathbf{x})\le\nu_0^\lambda\right\}. \end{cases}$$

Taking into account the symmetry of ψ^λ and ω^λ, we get

$$\begin{cases} \omega_{N+1}^\lambda = \lambda & \text{a.e. in} \quad B_{N+1}\cap\left\{\mathbf{x}\in D\mid \psi^\lambda(\mathbf{x})>\nu_0^\lambda\right\}, \\ \omega_{N+1}^\lambda = 0 & \text{a.e. in} \quad B_{N+1}\cap\left\{\mathbf{x}\in D\mid \psi^\lambda(\mathbf{x})\le\nu_0^\lambda\right\}, \end{cases} \tag{2.16}$$

or equivalently,

$$\omega_{N+1}^\lambda = \lambda I_{\left\{\mathbf{x}\in D|\psi^\lambda(\mathbf{x})>\nu_0^\lambda\right\}\cap B_{N+1}}.$$

By using strong maximum principles and the definition of ν_0^λ, we get $\nu_0^\lambda > 0$.

Finally, using similar methods we can also prove that there exists some $\nu_1^\lambda \in \mathbb{R}^+$ such that

$$\omega_i^\lambda = -\lambda I_{\left\{\mathbf{x}\in D|\psi^\lambda(\mathbf{x})<-\nu_1^\lambda\right\}\cap B_i},$$

for $i = 1, \cdots, N$. Thus, we complete the proof.

□

2.4. **Limiting behavior.** In this subsection, we analyze the limiting behavior of ω^λ as $\lambda \to +\infty$. To begin with, we need the following estimates for ω^λ. For convenience, we will use C to denote various positive numbers independent of λ in the rest of this section.

Lemma 2.6. *Let ω^λ be a maximizer obtained in Lemma 2.3. Then*

(1) $E(\omega^\lambda) \ge -\frac{N\kappa^2}{4\pi}\ln\varepsilon_1 - \frac{\mu^2}{4\pi}\ln\varepsilon_2 - C$, *where* $\varepsilon_1 = \sqrt{\frac{\kappa}{\lambda\pi}}$ *and* $\varepsilon_2 = \sqrt{\frac{\mu}{\lambda\pi}}$ *;*

(2) $\nu_0^\lambda \ge -\frac{\mu}{2\pi}\ln\varepsilon_2 + C$ *and* $\nu_1^\lambda \ge -\frac{\kappa}{2\pi}\ln\varepsilon_1 + C$*;*

(3) *diam*$(supp\omega_i^\lambda) \le R\varepsilon_1$ *for* $i = 1, \cdots, N$ *and diam*$(supp\omega_{N+1}^\lambda) \le R\varepsilon_2$, *where R is a positive number independent of λ;*

(4) *There holds*

$$\lim_{\lambda\to+\infty}\left| -\frac{1}{\kappa}\int_D \mathbf{x}\omega_i^\lambda(\mathbf{x})d\mathbf{x} - Q_i \right| = 0, \qquad \text{for} \quad i = 1, \cdots, N,$$

$$\lim_{\lambda\to+\infty}\left| \frac{1}{\mu}\int_D \mathbf{x}\omega_{N+1}^\lambda(\mathbf{x})d\mathbf{x} \right| = 0.$$

Proof. The proof is analogous to that in [28]. The key point is to estimate the total energy $E(\omega^\lambda)$ and the energy of the vortex core.

Note that $\pi\lambda\varepsilon_1^2=\kappa, \pi\lambda\varepsilon_2^2=\mu$. Firstly, taking test function $\tilde{\omega}^\lambda$ into $E(\omega)$, which is

$$\tilde{\omega}^\lambda=-\sum_{i=1}^{N}\lambda I_{B_{\varepsilon_1}(Q_i)}+\omega_{N+1}^\lambda:=\sum_{i=1}^{N+1}\tilde{\omega}_i^\lambda, \tag{2.17}$$

we get $\tilde{\omega}^\lambda\in M^\lambda(D)$ for λ sufficiently large. Therefore $E(\omega^\lambda)\geq E(\tilde{\omega}^\lambda)$, or equivalently,

$$\begin{aligned}
E(\omega^\lambda)\geq&\frac{1}{2}\int_D\int_D G(\mathbf{x},\mathbf{y})\sum_{i=1}^{N+1}\tilde{\omega}_i^\lambda(\mathbf{x})\sum_{j=1}^{N+1}\tilde{\omega}_j^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}\\
=&\frac{1}{2}\sum_{i=1}^{N+1}\int_D\int_D G(\mathbf{x},\mathbf{y})\tilde{\omega}_i^\lambda(\mathbf{x})\tilde{\omega}_i^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}+\frac{1}{2}\sum_{1\leq i\neq j\leq N+1}\int_D\int_D G(\mathbf{x},\mathbf{y})\tilde{\omega}_i^\lambda(\mathbf{x})\tilde{\omega}_j^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}\\
=&\frac{1}{2}\sum_{i=1}^{N}-\frac{1}{2\pi}\int_D\int_D\ln|\mathbf{x}-\mathbf{y}|\tilde{\omega}_i^\lambda(\mathbf{x})\tilde{\omega}_i^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}-\frac{1}{2}\sum_{i=1}^{N}\int_D\int_D h(\mathbf{x},\mathbf{y})\tilde{\omega}_i^\lambda(\mathbf{x})\tilde{\omega}_i^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}\\
&+\frac{1}{2}\sum_{1\leq i\neq j\leq N+1}\int_D\int_D G(\mathbf{x},\mathbf{y})\tilde{\omega}_i^\lambda(\mathbf{x})\tilde{\omega}_j^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}+\frac{1}{2}\int_D\int_D G(\mathbf{x},\mathbf{y})\tilde{\omega}_{N+1}^\lambda(\mathbf{x})\tilde{\omega}_{N+1}^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}\\
:=&I_1+I_2+I_3+\frac{1}{2}\int_D\int_D G(\mathbf{x},\mathbf{y})\omega_{N+1}^\lambda(\mathbf{x})\omega_{N+1}^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}.
\end{aligned} \tag{2.18}$$

Since $supp\tilde{\omega}_i^\lambda=\overline{B_{\varepsilon_1}(Q_i)}$, we get

$$I_1\geq\frac{1}{2}\sum_{i=1}^{N}-\frac{1}{2\pi}\ln 2\varepsilon_1\int_D\int_D\tilde{\omega}_i^\lambda(\mathbf{x})\tilde{\omega}_i^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}=-\frac{N\kappa^2}{4\pi}\ln\varepsilon_1+C. \tag{2.19}$$

By the choice of δ_0, we get

$$\begin{aligned}
|I_2|\leq&\frac{1}{2}\sum_{i=1}^{N}\sup_{\mathbf{x},\mathbf{y}\in suppB_{\delta_0}(Q_i)}|h(\mathbf{x},\mathbf{y})|\left|\int_D\tilde{\omega}_i^\lambda(\mathbf{x})d\mathbf{x}\right|\left|\int_D\tilde{\omega}_i^\lambda(\mathbf{y})d\mathbf{y}\right|\leq C,\\
|I_3|\leq&\frac{1}{2}\sum_{1\leq i\neq j\leq N+1}\sup_{\mathbf{x}\in B_{\delta_0}(Q_i),\mathbf{y}\in B_{\delta_0}(Q_j)}|G(\mathbf{x},\mathbf{y})|\left|\int_D\tilde{\omega}_i^\lambda(\mathbf{x})d\mathbf{x}\right|\left|\int_D\tilde{\omega}_i^\lambda(\mathbf{y})d\mathbf{y}\right|\leq C.
\end{aligned} \tag{2.20}$$

By combining (2.18), (2.19) and (2.20), we obtain

$$E(\omega^\lambda)\geq-\frac{N\kappa^2}{4\pi}\ln\varepsilon_1+C+\frac{1}{2}\int_D\int_D G(\mathbf{x},\mathbf{y})\omega_{N+1}^\lambda(\mathbf{x})\omega_{N+1}^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}. \tag{2.21}$$

Similarly, taking a test function $\tilde{\omega}^\lambda=-\sum_{i=1}^{N}\lambda I_{B_{\varepsilon_1}(Q_i)}+\lambda I_{B_{\varepsilon_2}(Q_{N+1})}$ into (2.17) and combing (2.18), (2.19) with (2.20), we can get

$$E(\omega^\lambda)\geq-\frac{N\kappa^2}{4\pi}\ln\varepsilon_1-\frac{\mu^2}{4\pi}\ln\varepsilon_2-C,$$

which proves (1).

To estimate ν_1^λ, we divide $E(\omega^\lambda)$ into two parts:

$$\begin{aligned} E(\omega^\lambda) =&\frac{1}{2}\int_D \psi^\lambda(\mathbf{x})\omega^\lambda(\mathbf{x})d\mathbf{x} = \frac{1}{2}\sum_{i=1}^{N+1}\int_D \psi^\lambda(\mathbf{x})\omega_i^\lambda(\mathbf{x})d\mathbf{x} \\ =&T_1(\omega^\lambda) + \frac{N\kappa}{2}\nu_1^\lambda + \frac{1}{2}\int_D \psi^\lambda(\mathbf{x})\omega_{N+1}^\lambda(\mathbf{x})d\mathbf{x}, \end{aligned} \tag{2.22}$$

where the energy of vortex core $T_1(\omega^\lambda)$ is defined as

$$T_1(\omega^\lambda) := \frac{1}{2}\sum_{i=1}^{N}\int_D \left(\psi^\lambda(\mathbf{x}) + \nu_1^\lambda\right)\omega_i^\lambda(\mathbf{x})d\mathbf{x}.$$

Note that

$$\begin{aligned} &\int_D \psi^\lambda(\mathbf{x})\omega_{N+1}^\lambda(\mathbf{x})d\mathbf{x} = \sum_{i=1}^{N+1}\int_D\int_D G(\mathbf{x},\mathbf{y})\omega_i^\lambda(\mathbf{x})\omega_{N+1}^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y} \\ &= \sum_{i=1}^{N}\int_D\int_D G(\mathbf{x},\mathbf{y})\omega_i^\lambda(\mathbf{x})\omega_{N+1}^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y} + \int_D\int_D G(\mathbf{x},\mathbf{y})\omega_{N+1}^\lambda(\mathbf{x})\omega_{N+1}^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y} \\ &\le C + \int_D\int_D G(\mathbf{x},\mathbf{y})\omega_{N+1}^\lambda(\mathbf{x})\omega_{N+1}^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}. \end{aligned} \tag{2.23}$$

Repeating the same argument as in the proof of lemma 3.5 in [14], we can get that

$$0 \le T_1(\omega^\lambda) = \frac{1}{2}\sum_{i=1}^{N}\int_D \left(\psi^\lambda(\mathbf{x}) + \nu_1^\lambda\right)\omega_i^\lambda(\mathbf{x})d\mathbf{x} \le C. \tag{2.24}$$

Combining this with (2.21), (2.22), (2.23) and (2.24), we can get

$$\nu_1^\lambda = \frac{2}{N\kappa}\left(E(\omega^\lambda) - T_1(\omega^\lambda) - \frac{1}{2}\int_D \psi^\lambda(\mathbf{x})\omega_{N+1}^\lambda(\mathbf{x})d\mathbf{x}\right) \ge -\frac{\kappa}{2\pi}\ln\varepsilon_1 + C. \tag{2.25}$$

Similarly, taking a test function $\tilde{\omega}^\lambda = \sum_{i=1}^N \omega_i^\lambda + \lambda I_{B_{\varepsilon_2}(Q_{N+1})}$ as in (2.17), we obtain

$$\nu_0^\lambda \ge -\frac{\mu}{2\pi}\ln\varepsilon_2 + C. \tag{2.26}$$

Now using the same idea as in [28] we can estimate the diameter of the support of each ω_i^λ. Without loss of generality, by the symmetry of ω^λ and ψ^λ, it suffices to prove the case of $i = N+1$. For any $\mathbf{x} \in supp\,\omega_{N+1}^\lambda$, by Lemma 2.5 we know $\psi^\lambda(\mathbf{x}) \ge \nu_0^\lambda$. By the definition of ψ^λ and (2.26), we get

$$\begin{aligned} &-\frac{\mu}{2\pi}\ln\varepsilon_2 + C \le \nu_0^\lambda \\ &\le -\frac{1}{2\pi}\int_D \ln|\mathbf{x}-\mathbf{y}|\omega_{N+1}^\lambda(\mathbf{y})d\mathbf{y} - \int_D h(\mathbf{x},\mathbf{y})\omega_{N+1}^\lambda(\mathbf{y})d\mathbf{y} + \sum_{i=1}^{N}\int_D G(\mathbf{x},\mathbf{y})\omega_i^\lambda(\mathbf{y})d\mathbf{y}. \end{aligned} \tag{2.27}$$

Since $h \in C^\infty(D \times D)$ and $G \in C^\infty(B_{N+1} \times B_i)$ for $i = 1, \cdots, N$, we have

$$\left| \int_D h(\mathbf{x}, \mathbf{y}) \omega_{N+1}^\lambda(\mathbf{y}) d\mathbf{y} \right| \leq \max_{\mathbf{x}, \mathbf{y} \in B_{N+1}} |h(\mathbf{x}, \mathbf{y})| \int_D \omega_{N+1}^\lambda(\mathbf{y}) d\mathbf{y} \leq C,$$
$$\left| \int_D G(\mathbf{x}, \mathbf{y}) \omega_i^\lambda(\mathbf{y}) d\mathbf{y} \right| \leq \max_{\mathbf{x} \in B_{N+1}, \mathbf{y} \in B_i} |G(\mathbf{x}, \mathbf{y})| \int_D -\omega_i^\lambda(\mathbf{y}) d\mathbf{y} \leq C.$$

Thus from (2.27), we get

$$-\frac{1}{2\pi} \int_D \ln |\mathbf{x} - \mathbf{y}| \omega_{N+1}^\lambda(\mathbf{y}) d\mathbf{y} \geq -\frac{\mu}{2\pi} \ln \varepsilon_2 + C,$$

namely,

$$\int_D \ln \frac{\varepsilon_2}{|\mathbf{x} - \mathbf{y}|} \omega_{N+1}^\lambda(\mathbf{y}) d\mathbf{y} \geq C. \tag{2.28}$$

On the other hand, for any fixed $R > 1$, by rearrangement inequality we get

$$\int_{B_{R\varepsilon_2}(\mathbf{x})} \ln \frac{\varepsilon_2}{|\mathbf{x} - \mathbf{y}|} \omega_{N+1}^\lambda(\mathbf{y}) d\mathbf{y} \leq \int_{B_{\varepsilon_2}(0)} \ln \frac{\varepsilon_2}{|\mathbf{x}|} \lambda d\mathbf{x} = \frac{\mu}{2}. \tag{2.29}$$

Combining (2.28) and (2.29), we obtain

$$C \leq \int_{D \backslash B_{R\varepsilon_2}(\mathbf{x})} \ln \frac{\varepsilon_2}{|\mathbf{x} - \mathbf{y}|} \omega_{N+1}^\lambda(\mathbf{y}) d\mathbf{y} \leq (-\ln R) \int_{D \backslash B_{R\varepsilon_2}(\mathbf{x})} \omega_{N+1}^\lambda(\mathbf{y}) d\mathbf{y},$$

which gives

$$\int_{D \backslash B_{R\varepsilon_2}(\mathbf{x})} \omega_{N+1}^\lambda(\mathbf{y}) d\mathbf{y} \leq \frac{C}{\ln R}.$$

Taking $R > 1$ sufficiently large, we have for any $\mathbf{x} \in supp\omega_{N+1}^\lambda$

$$\int_{D \backslash B_{R\varepsilon_2}(\mathbf{x})} \omega_{N+1}^\lambda(\mathbf{y}) d\mathbf{y} \leq \frac{\mu}{4}, \tag{2.30}$$

thus for any $\mathbf{x} \in supp\omega_{N+1}^\lambda$, $\int_{B_{R\varepsilon_2}(\mathbf{x})} \omega_{N+1}^\lambda(\mathbf{y}) d\mathbf{y} \geq 3\mu/4$. Using the same calculation as in [28], we can get

$$diam(supp\omega_{N+1}^\lambda) \leq 2R\varepsilon_2. \tag{2.31}$$

By similar methods we can also get $diam(supp\omega_i^\lambda) \leq 2R\varepsilon_1$ for $i = 1, \cdots, N$.

From now on, we have proved (3) of Lemma 3.4, which shows that the support of each piece of vorticity ω^λ concentrates on some point as λ tends to infinity. Since $supp\omega_i^\lambda(\mathbf{x}) \subset \overline{B_i}$, we obtain that up to a subsequence

$$\begin{aligned} &-\frac{1}{\kappa} \lim_{\lambda \to +\infty} \int_D \mathbf{x} \omega_i^\lambda(\mathbf{x}) d\mathbf{x} = \bar{\mathbf{x}}_i, \qquad \text{for } \ i = 1, \cdots, N, \\ &\frac{1}{\mu} \lim_{\lambda \to +\infty} \int_D \mathbf{x} \omega_{N+1}^\lambda(\mathbf{x}) d\mathbf{x} = \bar{\mathbf{x}}_{N+1}. \end{aligned} \tag{2.32}$$

for some $\bar{\mathbf{x}}_i \in \overline{B_i}$. Since $\mathcal{R}_i(\omega^\lambda) = \omega^\lambda$, we obtain $\bar{\mathbf{x}}_{N+1} = \mathbf{0}$. Moreover, since $\mathbf{e}^{\mathbf{i}\frac{2\pi(i-1)}{N}}(\omega^\lambda) = \omega^\lambda$ and $\mathcal{R}_i(\omega^\lambda) = \omega^\lambda$, we obtain $\bar{\mathbf{x}}_i = \rho_0\left(\cos\frac{2\pi(i-1)}{N}, \sin\frac{2\pi(i-1)}{N}\right)$ for some $\rho_0 \in (0,1)$, $i = 1, \cdots, N$.

In the following, we show that $(\bar{\mathbf{x}}_1, \cdots, \bar{\mathbf{x}}_{N+1}) \in D^{(N+1)}$ is a minimum point of the Kirchhoff-Routh function H_{N+1} on the set $\mathcal{S}_{N+1} \cap \left\{(\mathbf{x}_1, \cdots, \mathbf{x}_{N+1}) \in D^{(N+1)} \mid \mathbf{x}_i \in \overline{B_i}\right\}$. Thus combining with Lemma 2.1, we can easily deduce $\bar{\mathbf{x}}_i = Q_i$ for $i = 1, \cdots, N+1$. In fact, for each $\mathbf{z}_0 \in B_1$ such that $\mathbf{z}_0 = \rho(1,0)$ for some $\rho \in (0,1)$, we take the following test function

$$\bar{\omega}^\lambda = -\lambda \sum_{i=1}^{N} I_{B_{\varepsilon_1}\left(\mathbf{e}^{\mathbf{i}\frac{2\pi(i-1)}{N}}(\mathbf{z}_0)\right)} + \lambda I_{B_{\varepsilon_2}(\mathbf{0})} := \sum_{i=1}^{N+1} \bar{\omega}_i^\lambda.$$

It is easy to check that $\bar{\omega}^\lambda(x) \in M^\lambda(D)$ for λ sufficiently large. Therefore $E(\omega^\lambda) \geq E(\bar{\omega}^\lambda)$, or equivalently,

$$\int_D\int_D G(\mathbf{x},\mathbf{y}) \sum_{i=1}^{N+1} \bar{\omega}_i^\lambda(\mathbf{x}) \sum_{j=1}^{N+1} \bar{\omega}_j^\lambda(\mathbf{y}) d\mathbf{x}d\mathbf{y} \leq \int_D\int_D G(\mathbf{x},\mathbf{y}) \sum_{i=1}^{N+1} \omega_i^\lambda(\mathbf{x}) \sum_{j=1}^{N+1} \omega_j^\lambda(\mathbf{y}) d\mathbf{x}d\mathbf{y}. \tag{2.33}$$

On the one hand, by using the rearrangement inequality we have

$$\begin{aligned}
&\int_D\int_D G(\mathbf{x},\mathbf{y}) \sum_{i=1}^{N+1} \omega_i^\lambda(\mathbf{x}) \sum_{j=1}^{N+1} \omega_j^\lambda(\mathbf{y}) d\mathbf{x}d\mathbf{y} \\
&= \sum_{i=1}^{N+1} \int_D\int_D G(\mathbf{x},\mathbf{y})\omega_i^\lambda(\mathbf{x})\omega_i^\lambda(\mathbf{y}) d\mathbf{x}d\mathbf{y} + \sum_{1\leq i\neq j\leq N+1} \int_D\int_D G(\mathbf{x},\mathbf{y})\omega_i^\lambda(\mathbf{x})\omega_j^\lambda(\mathbf{y}) d\mathbf{x}d\mathbf{y} \\
&= \sum_{i=1}^{N+1} -\frac{1}{2\pi} \int_D\int_D \ln|\mathbf{x}-\mathbf{y}|\omega_i^\lambda(\mathbf{x})\omega_i^\lambda(\mathbf{y}) d\mathbf{x}d\mathbf{y} - \sum_{i=1}^{N+1} \int_D\int_D h(\mathbf{x},\mathbf{y})\omega_i^\lambda(\mathbf{x})\omega_i^\lambda(\mathbf{y}) d\mathbf{x}d\mathbf{y} \\
&+ \sum_{1\leq i\neq j\leq N+1} \int_D\int_D G(\mathbf{x},\mathbf{y})\omega_i^\lambda(\mathbf{x})\omega_j^\lambda(\mathbf{y}) d\mathbf{x}d\mathbf{y} \\
&\leq \sum_{i=1}^{N+1} -\frac{1}{2\pi} \int_D\int_D \ln|\mathbf{x}-\mathbf{y}|\bar{\omega}_i^\lambda(\mathbf{x})\bar{\omega}_i^\lambda(\mathbf{y}) d\mathbf{x}d\mathbf{y} - \sum_{i=1}^{N+1} \int_D\int_D h(\mathbf{x},\mathbf{y})\omega_i^\lambda(\mathbf{x})\omega_i^\lambda(\mathbf{y}) d\mathbf{x}d\mathbf{y} \\
&+ \sum_{1\leq i\neq j\leq N+1} \int_D\int_D G(\mathbf{x},\mathbf{y})\omega_i^\lambda(\mathbf{x})\omega_j^\lambda(\mathbf{y}) d\mathbf{x}d\mathbf{y}.
\end{aligned} \tag{2.34}$$

On the other hand, it follows from the decomposition of G that

$$\begin{aligned}&\int_D\int_D G(\mathbf{x},\mathbf{y})\sum_{i=1}^{N+1}\bar{\omega}_i^\lambda(\mathbf{x})\sum_{j=1}^{N+1}\bar{\omega}_j^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}\\ &=\sum_{i=1}^{N+1}-\frac{1}{2\pi}\int_D\int_D \ln|\mathbf{x}-\mathbf{y}|\bar{\omega}_i^\lambda(\mathbf{x})\bar{\omega}_i^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}-\sum_{i=1}^{N+1}\int_D\int_D h(\mathbf{x},\mathbf{y})\bar{\omega}_i^\lambda(\mathbf{x})\bar{\omega}_i^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}\\ &+\sum_{1\le i\neq j\le N+1}\int_D\int_D G(\mathbf{x},\mathbf{y})\bar{\omega}_i^\lambda(\mathbf{x})\bar{\omega}_j^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}.\end{aligned}\tag{2.35}$$

Combining (2.33), (2.34) and (2.35), we get

$$\begin{aligned}&\sum_{1\le i\neq j\le N+1}\int_D\int_D G(\mathbf{x},\mathbf{y})\bar{\omega}_i^\lambda(\mathbf{x})\bar{\omega}_j^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}-\sum_{i=1}^{N+1}\int_D\int_D h(\mathbf{x},\mathbf{y})\bar{\omega}_i^\lambda(\mathbf{x})\bar{\omega}_i^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}\\ &\le\sum_{1\le i\neq j\le N+1}\int_D\int_D G(\mathbf{x},\mathbf{y})\omega_i^\lambda(\mathbf{x})\omega_j^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}-\sum_{i=1}^{N+1}\int_D\int_D h(\mathbf{x},\mathbf{y})\omega_i^\lambda(\mathbf{x})\omega_i^\lambda(\mathbf{y})d\mathbf{x}d\mathbf{y}.\end{aligned}\tag{2.36}$$

Using (2.31), (2.32) and taking λ to infinity, we get

$$-H_{N+1}\left(\mathbf{z}_0,\mathbf{e}^{\mathbf{i}\frac{2\pi}{N}}(\mathbf{z}_0),\cdots,\mathbf{e}^{\mathbf{i}\frac{2\pi(N-1)}{N}}(\mathbf{z}_0),\mathbf{0}\right)\le -H_{N+1}\left(\bar{\mathbf{x}}_1,\bar{\mathbf{x}}_2,\cdots,\bar{\mathbf{x}}_{N+1}\right)\tag{2.37}$$

for any $\mathbf{z}_0\in\{\mathbf{z}\in\overline{B_1}\mid \mathbf{z}=\rho(1,0),\rho\in(0,1)\}$. By (2.37), we obtain that $(\bar{\mathbf{x}}_1,\bar{\mathbf{x}}_2,\cdots,\bar{\mathbf{x}}_{N+1})$ is a minimum point of H_{N+1} on $\mathcal{S}_{N+1}\cap\left\{(\mathbf{x}_1,\cdots,\mathbf{x}_{N+1})\in D^{(N+1)}\mid \mathbf{x}_i\in\overline{B_i}\right\}$. Thus we get $\rho_0=\tilde{\rho}$ by Lemma 2.1. Namely, $\bar{\mathbf{x}}_i=Q_i$ for $i=1,\cdots,N+1$. □

2.5. **Proof of Theorem 1.2.** Before proving Theorem 1.2, we give a lemma from [11], which is a criterion for weak solutions of (1.6).

Lemma A. *Let $\omega\in L^{\frac{4}{3}}(D)$. Suppose that $\omega=f(\psi)$ a.e. in D for some monotonic function $f:\mathbb{R}\to\mathbb{R}$, then ω is a weak solution of* (1.6).

Now, we are able to prove Theorem 1.2. The basic idea is to show that ω^λ satisfies the condition in Lemma A if λ is sufficiently large.

Proof of Theorem 1.2. It suffices to show that ω^λ is a weak solution of (1.6). First we show that

$$\sup_{\partial B_1}|\psi^\lambda|\le C.\tag{2.38}$$

By Lemma 2.1 and Lemma 2.6, for $\delta_2=\frac{\delta_0}{2}$, there exists $\lambda_0>0$ such that for any $\lambda>\lambda_0$,

$$dist\left(supp\omega_1^\lambda,\partial B_1\right)>\delta_2.$$

Thus for each $\mathbf{x} \in \partial B_1$,

$$
\begin{aligned}
|\psi^\lambda(\mathbf{x})| =& \left| \int_D G(\mathbf{x},\mathbf{y}) \sum_{i=1}^{N+1} \omega_i^\lambda(\mathbf{y}) d\mathbf{y} \right| \\
=& \left| -\frac{1}{2\pi} \int_D \ln|\mathbf{x}-\mathbf{y}| \omega_1^\lambda(\mathbf{y}) d\mathbf{y} + \int_D h(\mathbf{x},\mathbf{y}) \omega_1^\lambda(\mathbf{y}) d\mathbf{y} + \int_D G(\mathbf{x},\mathbf{y}) \sum_{i=2}^{N+1} \omega_i^\lambda(\mathbf{y}) d\mathbf{y} \right| \\
\leq& \left| -\frac{1}{2\pi} \int_D \ln|\mathbf{x}-\mathbf{y}| \omega_1^\lambda(\mathbf{y}) d\mathbf{y} \right| + \left| \int_D h(\mathbf{x},\mathbf{y}) \omega_1^\lambda(\mathbf{y}) d\mathbf{y} \right| + \left| \int_D G(\mathbf{x},\mathbf{y}) \sum_{i=2}^{N+1} \omega_i^\lambda(\mathbf{y}) d\mathbf{y} \right| \\
\leq& \frac{1}{2\pi} \left| \int_D \omega_1^\lambda(\mathbf{y}) d\mathbf{y} \cdot \ln \delta_2 \right| + \max_{\mathbf{x}\in\partial B_1, \mathbf{y}\in supp\omega_1^\lambda} |h(\mathbf{x},\mathbf{y})| \cdot \left| \int_D \omega_1^\lambda(\mathbf{y}) d\mathbf{y} \right| \\
&+ \max_{\mathbf{x}\in\partial B_1, \mathbf{y}\in\cup_{i=2}^{N+1} B_i} |G(\mathbf{x},\mathbf{y})| \cdot \sum_{i=2}^{N+1} \left| \int_D \omega_i^\lambda(\mathbf{y}) d\mathbf{y} \right| \\
\leq& -\frac{\kappa \ln \delta_2}{2\pi} + C,
\end{aligned}
$$

where we use the regularity of G and h and the fact that the integral of each ω_i^λ is $-\kappa$ and μ.

Similarly, we can prove that $|\psi^\lambda| \leq C$ on $\cup_{i=2}^{N+1} \partial B_i$. Thus ψ^λ is harmonic in $D \setminus \overline{\cup_{i=1}^{N+1} B_i}$ and by the maximum principle it is easy to deduce that

$$|\psi^\lambda| \leq C \qquad \text{in} \;\; D \setminus \cup_{i=1}^{N+1} B_i.$$

Combining (2) in Lemma 2.6 we get that, there exists $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$, there holds

$$
\begin{aligned}
&\left\{ \mathbf{x} \in D \mid \psi^\lambda(\mathbf{x}) > \nu_0^\lambda \right\} \cap B_{N+1} = \left\{ \mathbf{x} \in D \mid \psi^\lambda(\mathbf{x}) > \nu_0^\lambda \right\}, \\
&\left\{ \mathbf{x} \in D \mid \psi^\lambda(\mathbf{x}) < -\nu_1^\lambda \right\} \cap \bigcup_{i=1}^{N} B_i = \left\{ \mathbf{x} \in D \mid \psi^\lambda(\mathbf{x}) < -\nu_1^\lambda \right\}.
\end{aligned} \tag{2.39}
$$

Recall that by Lemma 2.3 ω^λ has the following form

$$\omega^\lambda = \lambda I_{\left\{\mathbf{x}\in D | \psi^\lambda(\mathbf{x}) > \nu_0^\lambda\right\} \cap B_{N+1}} - \lambda \sum_{i=1}^{N} I_{\left\{\mathbf{x}\in D | \psi^\lambda(\mathbf{x}) < -\nu_1^\lambda\right\} \cap B_i}.$$

Taking into account (2.39) we get for each $\lambda > \lambda_0$,

$$\omega^\lambda = \lambda I_{\left\{\mathbf{x}\in D | \psi^\lambda(\mathbf{x}) > \nu_0^\lambda\right\}} - \lambda I_{\left\{\mathbf{x}\in D | \psi^\lambda(\mathbf{x}) < -\nu_1^\lambda\right\}}.$$

Applying lemma A, we get that ω^λ is a weak solution to (1.6), which completes the proof of Theorem 1.2.

□

3. Proof of Theorem 1.3

In this section, we give the proof Theorem 1.3 by solving a variational problem for the vorticity and studying the limiting behavior. Throughout this section, we will choose $k = 2N$ in (1.9), where N is a given positive integer. We also assume that $\kappa_i = (-1)^{i-1}$ for each $i = 1, \cdots, 2N$. The construction is similar to the one for Theorem 1.2, so we only sketch the proof and emphasize the different parts.

Let us start with the minimization of H_{2N} on the following subset of $D^{(2N)}$:

$$\mathcal{V}_{2N} := \left\{(\mathbf{x}_1, \cdots, \mathbf{x}_{2N}) \in D^{(2N)} \mid \mathbf{x}_i = \rho\left(\cos(\frac{\pi(i-1)}{N}), \sin(\frac{\pi(i-1)}{N})\right),\ 1 \le i \le 2N,\ \rho \in (0,1)\right\}.$$

We have

Lemma 3.1. *There exists an isolated minimum point $(P_1, \cdots, P_{2N})$ for H_{2N} on $\mathcal{V}_{2N}$.*

Proof. For any $(\mathbf{x}_1, \cdots, \mathbf{x}_{2N}) \in \mathcal{V}_{2N}$ with $\mathbf{x}_i = \rho\left(\cos(\frac{\pi(i-1)}{N}), \sin(\frac{\pi(i-1)}{N})\right)$ for $i = 1, \cdots, 2N$, by the symmetry of the Green's function, the Kirchhoff-Routh function H_{2N} can be written as

$$\begin{aligned} H_{2N}(\mathbf{x}_1, \cdots, \mathbf{x}_{2N}) =& -2N\sum_{i=2}^{N} G(\mathbf{x}_1, \mathbf{x}_{2i-1}) + 2N\sum_{i=1}^{N} G(\mathbf{x}_1, \mathbf{x}_{2i}) + 2Nh(\mathbf{x}_1, \mathbf{x}_1) \\ :=& B_1 + B_2 + B_3. \end{aligned}$$

Note that

$$G(\mathbf{x}_i, \mathbf{x}_j) = \frac{1}{2\pi}\ln\frac{|\mathbf{x}_j|\left|\mathbf{x}_i - \frac{\mathbf{x}_j}{|\mathbf{x}_j|^2}\right|}{|\mathbf{x}_i - \mathbf{x}_j|} = \frac{1}{2\pi}\ln\frac{\rho\sqrt{\rho^2 + \frac{1}{\rho^2} - 2\cos\frac{\pi(i-j)}{N}}}{2\rho\sin\frac{\pi(i-j)}{2N}},\ h(\mathbf{x}_i, \mathbf{x}_i) = \frac{1}{2\pi}\ln\frac{1}{1-\rho^2},$$

so we get

$$\begin{aligned} B_1 =& -\frac{2N}{4\pi}\sum_{i=2}^{N}\ln\frac{\rho^4 + 1 - 2\rho^2\cos\frac{2\pi(2i-2)}{2N}}{4\rho^2\sin^2\frac{\pi(2i-2)}{2N}} = -\frac{N}{2\pi}\sum_{i=2}^{N}\ln\frac{\rho^4 + 1 - 2\rho^2\cos\frac{2\pi(i-1)}{N}}{4\rho^2\sin^2\frac{\pi(i-1)}{N}}, \\ B_2 =& \frac{2N}{4\pi}\sum_{i=1}^{N}\ln\frac{\rho^4 + 1 - 2\rho^2\cos\frac{2\pi(2i-1)}{2N}}{4\rho^2\sin^2\frac{\pi(2i-1)}{2N}} = \frac{N}{2\pi}\sum_{i=1}^{N}\ln\frac{\rho^4 + 1 - 2\rho^2\cos\frac{\pi(2i-1)}{N}}{4\rho^2\sin^2\frac{\pi(2i-1)}{2N}}, \\ B_3 =& \frac{N}{\pi}\ln\frac{1}{1-\rho^2}. \end{aligned}$$

By direct calculation, we have

$$\begin{aligned} H_{2N}(\rho) &= B_1 + B_2 + B_3 \\ &= \frac{N}{2\pi}\ln\left[\prod_{i=2}^{N}\frac{4\rho^2\sin^2\frac{\pi(i-1)}{N}}{\rho^4 + 1 - 2\rho^2\cos\frac{2\pi(i-1)}{N}}\prod_{i=1}^{N}\frac{\rho^4 + 1 - 2\rho^2\cos\frac{\pi(2i-1)}{N}}{4\rho^2\sin^2\frac{\pi(2i-1)}{2N}}\frac{1}{(1-\rho^2)^2}\right] \\ &:= \frac{N}{2\pi}\ln g(\rho), \end{aligned}$$

where

$$\begin{aligned} g(\rho) =& \prod_{i=2}^{N} \frac{4\rho^2 \sin^2 \frac{\pi(i-1)}{N}}{\rho^4 + 1 - 2\rho^2 \cos \frac{2\pi(i-1)}{N}} \prod_{i=1}^{N} \frac{\rho^4 + 1 - 2\rho^2 \cos \frac{\pi(2i-1)}{N}}{4\rho^2 \sin^2 \frac{\pi(2i-1)}{2N}} \frac{1}{(1-\rho^2)^2} \\ =& \frac{\prod_{i=1}^{N}(\rho^4 + 1 - 2\rho^2 \cos \frac{\pi(2i-1)}{N})}{\prod_{i=2}^{N}(\rho^4 + 1 - 2\rho^2 \cos \frac{2\pi(i-1)}{N})\rho^2(1-\rho^2)^2} \cdot \frac{\prod_{i=2}^{N} \sin^2 \frac{\pi(i-1)}{N}}{4\prod_{i=1}^{N} \sin^2 \frac{\pi(2i-1)}{2N}}. \end{aligned} \tag{3.1}$$

Since $\cos \frac{\pi(2i-1)}{N} < 1$ for each $i = 1, \cdots, N$ and $\cos \frac{2\pi(i-1)}{N} < 1$ for each $i = 2, \cdots, N$, we know that there exists $\bar{\delta} > 0$, such that $\rho^4 + 1 - 2\rho^2 \cos \frac{\pi(2i-1)}{N} \geq \bar{\delta}$ and $\rho^4 + 1 - 2\rho^2 \cos \frac{2\pi(i-1)}{N} \geq \bar{\delta}$ for any $\rho \in (0, 1)$. Taking into account (3.1), we get

$$\lim_{\rho \to 0^+} g(\rho) = +\infty, \ \lim_{\rho \to 1^-} g(\rho) = +\infty.$$

On the other hand, since g is continuous and nonnegative in $(0, 1)$, g attains its minimum in $(0, 1)$. Since the numerator and denominator of $g(\rho)$ are both polynomials, it is easy to check that each minimum point of g must be isolated. Therefore, there must exist a $\bar{\rho} \in (0, 1)$ being an isolated minimum point of $H_{2N}(\rho)$. Let $P_i := \bar{\rho}\left(\cos \frac{\pi(i-1)}{N}, \sin \frac{\pi(i-1)}{N}\right)$ for $i = 1, \cdots, 2N$, then $(P_1, \cdots, P_{2N})$ is an isolated minimum point of $H_{2N}(\mathbf{x}_1, \cdots, \mathbf{x}_{2N})$ on $\mathcal{V}_{2N}$.

□

Let $(P_1, \cdots, P_{2N})$ be an isolated minimum point of H_{2N} obtained in Lemma 3.1. Since $P_i \neq P_j$ for $i \neq j$, we can choose a sufficiently small $\delta_0 > 0$ such that $B_{\delta_0}(P_i) \subset\subset D$ for $i = 1, \cdots, 2N$ and $\overline{B_{\delta_0}(P_i)} \cap \overline{B_{\delta_0}(P_j)} = \varnothing$ for $i \neq j$. Moreover, since $(P_1, \cdots, P_{2N})$ is an isolated minimum point of H_{2N}, by choosing a smaller δ_0, we assume that $(P_1, \cdots, P_{2N})$ is a unique minimum point on $\mathcal{V}_{2N} \cap \left\{(\mathbf{x_1}, \cdots, \mathbf{x_{2N}}) \in D^{(2N)} : \mathbf{x_i} \in \overline{B_{\delta_0}(P_i)}\right\}$.

Define the following admissible class

$$\begin{aligned} K_\lambda(D) = \Bigg\{ & \omega \in L^\infty(D) \mid \omega = \sum_{i=1}^{2N} \omega_i, supp\omega_i \subset B_{\delta_0}(P_i), \\ & \omega_i = \omega I_{B_{\delta_0}(P_i)}, \ \int_D \omega_1(\mathbf{x}) d\mathbf{x} = 1, \ 0 \leq \omega_1 \leq \lambda \ \text{ a.e. in } \ D, \\ & \mathbf{e}^{\mathbf{i}\frac{\pi(i-1)}{N}}(\omega) = (-1)^{i-1}\omega \text{ and } \mathcal{T}_i(\omega) = \omega \ \text{ for } i = 1, \cdots, 2N \Bigg\}. \end{aligned}$$

Note also that $K_\lambda(D)$ is not empty when λ is sufficiently large.

Now we consider the maximization of $E(\omega)$ on the admissible class $K_\lambda(D)$. By exactly the same as the one in Lemma 2.3 we can prove

Lemma 3.2. *There exists $\omega^\lambda \in K_\lambda(D)$, such that $E(\omega^\lambda) = \sup_{\omega \in K_\lambda(D)} E(\omega)$.*

Similar to Lemma 2.5, we can also obtain the profile of ω^λ obtained in Lemma 3.2.

Lemma 3.3. *There exists $\tau^\lambda \in \mathbb{R}^+$ such that for each $i = 1, \cdots, N$,*

$$\begin{aligned} \omega_{2i-1}^\lambda &= \lambda I_{\left\{\mathbf{x}\in D|\psi^\lambda(\mathbf{x})>\tau^\lambda\right\}\cap B_{\delta_0}(P_{2i-1})}, \\ \omega_{2i}^\lambda &= -\lambda I_{\left\{\mathbf{x}\in D|\psi^\lambda(\mathbf{x})<-\tau^\lambda\right\}\cap B_{\delta_0}(P_{2i})}. \end{aligned} \tag{3.2}$$

As in Lemma 2.6, we can obtain the following limiting behavior of ω^λ as $\lambda \to +\infty$.

Lemma 3.4. *Let $\varepsilon = \sqrt{\frac{1}{\pi\lambda}}$. Then*

(1) $E(\omega^\lambda) \geq -\frac{N}{\pi}\ln\varepsilon - C$;
(2) $\tau^\lambda \geq -\frac{1}{2\pi}\ln\varepsilon - C$;
(3) there exists $R > 0$ not depending on λ, such that $diam(supp\omega_i^\lambda) \leq R\varepsilon$ for $i = 1, \cdots, 2N$;
(4) for each $i = 1, \cdots, 2N$, we have

$$\lim_{\lambda\to+\infty}\int_D \mathbf{x}\omega_i^\lambda(\mathbf{x})d\mathbf{x} = P_i, \tag{3.3}$$

where C is used to denote various constants independent of λ.

Remark 3.5. By (3) and (4) in Lemma 3.4 it is easy to check that for λ sufficiently large,

$$\begin{aligned} &\left\{\mathbf{x}\in D \mid \psi^\lambda(\mathbf{x}) > \tau^\lambda\right\} \cap B_{\delta_0}(P_{2i-1}) \subset\subset B_{\delta_0}(P_{2i-1}), \\ &\left\{\mathbf{x}\in D \mid \psi^\lambda(\mathbf{x}) < -\tau^\lambda\right\} \cap B_{\delta_0}(P_{2i}) \subset\subset B_{\delta_0}(P_{2i}), \end{aligned} \tag{3.4}$$

for each $i = 1, \cdots, N$. In other words, the support of ω_i^λ is strictly contained in $B_{\delta_0}(P_i)$.

Now we are ready to prove Theorem 1.3.

Similar as the proof of Theorem 1.2, it suffices to show that ω^λ is a steady vortex patch. By Lemma 3.4 and using the same analysis as in proof of Theorem 1.2, we get that there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$, there holds

$$\begin{aligned} &\left\{\mathbf{x}\in D \mid \psi^\lambda(\mathbf{x}) > \tau^\lambda\right\} \cap B_{\delta_0}(P_{2i-1}) = \left\{\mathbf{x}\in D \mid \psi^\lambda(\mathbf{x}) > \tau^\lambda\right\}, \\ &\left\{\mathbf{x}\in D \mid \psi^\lambda(\mathbf{x}) < -\tau^\lambda\right\} \cap B_{\delta_0}(P_{2i}) = \left\{\mathbf{x}\in D \mid \psi^\lambda(\mathbf{x}) < -\tau^\lambda\right\}. \end{aligned} \tag{3.5}$$

Moreover by Lemma 3.3, we obtain that ω^λ has the following form:

$$\omega^\lambda = \lambda\sum_{i=1}^N I_{\left\{\mathbf{x}\in D|\psi^\lambda(\mathbf{x})>\tau^\lambda\right\}\cap B_{\delta_0}(P_{2i-1})} - \lambda\sum_{i=1}^N I_{\left\{\mathbf{x}\in D|\psi^\lambda(\mathbf{x})<-\tau^\lambda\right\}\cap B_{\delta_0}(P_{2i})}.$$

Thus we obtain that for $\lambda > \lambda_0$, there holds

$$\omega^\lambda = \lambda I_{\left\{\mathbf{x}\in D|\psi^\lambda(\mathbf{x})>\tau^\lambda\right\}} - \lambda I_{\left\{\mathbf{x}\in D|\psi^\lambda(\mathbf{x})<-\tau^\lambda\right\}}.$$

By Lemma A, we finish the proof of Theorem 1.3.

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