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Vector-valued Sobolev spaces based on Banach function spaces[☆]

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ABSTRACT

It is known that there are several approaches to define a Sobolev class for Banach valued functions. We compare the usual definition via weak derivatives with the Reshetnyak–Sobolev space and with the Newtonian space; in particular, we provide sufficient conditions when all three agree. Also, we revise the difference quotient criterion and the property of Lipschitz mapping to preserve Sobolev space when it is acting as a superposition operator.

1. Introduction

Our primary motivation behind this work is to provide a non-differential characterization of Sobolev spaces. In particular, this would supply us with tools for analysing functions valued in a family of Banach spaces, e.g. [7]. Such functions typically appear in the theory of evolution PDEs. The other side of the work is that we consider Sobolev type spaces built upon a general Banach function norm.

A general idea of our study is to make use of analysis in metric spaces but taking into account the presence of a linear structure. The theory of Sobolev spaces on metric measure spaces is quite developed now. For a detailed treatment and for references to the literature on the subject, one may refer to the [11] by J. Heinonen, [9] by P. Hajłasz and P. Koskela, and [12] by J. Heinonen, P. Koskela, N. Shanmugalingam and J.T. Tyson.

In the present paper, we study the Sobolev space of vector-valued functions $W^1X(\Omega;V)$ based on a Banach function space $X(\Omega)$, where $\Omega\subset\mathbb{R}^n$. We discuss the connection W^1X with the Newtonian space

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N^1X and with the Reshetnyak–Sobolev space R^1X and provide sufficient conditions when $W^1X = R^1X = N^1X$. More precisely, we prove that $W^1X = R^1X$ if and only if V has the Radon–Nikodým property, whereas $R^1X = N^1X$ whenever the Meyers–Serrin theorem holds true for $W^1X(\Omega;\mathbb{R})$. Besides, we provide the difference quotient criterion and, as a consequence, obtain a version of pointwise description for Sobolev functions. Finally, we consider a question when Lipschitz mapping $f : V \to Z$ preserves a Sobolev class. It is always the case for R^1X and N^1X, while we should assume that Z enjoys the Radon–Nikodým property to have inclusion $f(W^1X(\Omega;V)) \subset W^1X(\Omega;Z)$. In particular, this means that the nonlinear superposition operator $N_f u = f \circ u$ is correctly defined.

We briefly discuss the limitations of the present work. In this study, functions are defined on a domain Ω in $\mathbb{R}^n$, and this provides us an opportunity to compare the weak gradient with upper gradients. In principle, all the introduced classes but $W^1X(\Omega;V)$ could be considered in metric settings. As we do not prove embedding theorems, we do not require any specific regularity of the domain Ω. On the other hand, embedding theorems could be recovered from the results for scalar functions (see Theorem 4.6). Finally, we point out that we consider a general Banach function space, and do not treat any specific examples, such as Orlicz or Lorentz spaces. At this stage, most of the methods are extensions of ones from the theory of L^p-spaces.

It happened that merely in the same time I. Caamaño, J. A. Jaramillo, Á. Prieto, and A. Ruiz in [5] did the research which partly intersects with ours.

2. Preliminaries

Throughout the paper $\Omega \subset \mathbb{R}^n$ and $|\cdot|$ denotes n-dimensional Lebesgue measure, and we use $\mu^{n-1}(\cdot)$ for $(n-1)$-dimensional Lebesgue measure. It should not cause any ambiguity that the modulus of a real number is also denoted via $|\cdot|$.

Let $M(\Omega)$ be the set of all real-valued measurable functions on Ω. A Banach space $X(\Omega)$ is said to be a *Banach function space* if it satisfies the following conditions:

(P1) if $|f| \le g$ a.e. with $f \in M(\Omega)$ and $g \in X(\Omega)$, then $f \in X(\Omega)$ and $\|f\|_{X(\Omega)} \le \|g\|_{X(\Omega)}$ (the lattice property);

(P2) if $0 \le f_n \nearrow f$ a.e., then $\|f_n\|_{X(\Omega)} \nearrow \|f\|_{X(\Omega)}$ (the Fatou property);

(P3) for any measurable set $A \subset \Omega$ with $|A| < \infty$, we have $\chi_A \in X(\Omega)$;

(P4) for any measurable set $A \subset \Omega$ with $|A| < \infty$, there exists a positive constant C_A such that $\|f\|_{L^1(A)} \le C_A \|f \cdot \chi_A\|_{X(\Omega)}$ for all $f \in X(\Omega)$.

When there is no ambiguity, we write $\|\cdot\|_X$ for $\|\cdot\|_{X(\Omega)}$.

Here we collect some notions and properties from the theory of Banach function spaces that are necessary. For a comprehensive exposition of the theory, we refer the reader to book [20].

Let $\{A_n\}$ be a sequence of measurable subsets of Ω, we say $A_n \to \emptyset$ if $\chi_{A_n} \to 0$ a.e. on Ω. Function space $X(\Omega)$ has *absolutely continuous norm* if $\|f \cdot \chi_{A_n}\|_{X(\Omega)} \to 0$ whenever $A_n \to \emptyset$ for any $f \in X(\Omega)$. (Examples L^p $(1 \le p < \infty)$, Lorentz $L^{p,q}$ $(1 \le q < \infty)$, see [20, p. 216].)

Define *the translation operator* τ_h, with $h \in \mathbb{R}^n$ for $u \in M(\Omega)$ by

$$\tau_h u(x) = \begin{cases} u(x+h), & \text{if } x+h \in \Omega, \\ 0, & \text{if } x+h \notin \Omega. \end{cases}$$

We say that $\|\cdot\|_{X(\Omega)}$ has *the translation inequality property* if for all $u \in X(\Omega)$ and all $h \in \mathbb{R}^n$ $\|\tau_h u\|_X \le \|u\|_X$. Note that every rearrangement invariant function norm possesses the translation inequality property.

Let $X(\Omega)$ be a Banach function space and let $X'(\Omega)$ be its associate space. Then, for functions $u \in X(\Omega)$ and $v \in X'(\Omega)$ the following Hölder inequality holds

$$\int_\Omega |uv|\, dx \le \|u\|_X \|v\|_{X'},$$

see [20, Theorem 6.2.6]. We will need the following Fatou lemma for Banach function spaces.

Lemma 2.1 *([20, Lemma 6.1.12]). Let $X(\Omega)$ be a Banach function space and assume that $f_n \in X(\Omega)$ and $f_n \to f$ a.e. on Ω for some $f \in M(\Omega)$. Assume further that*

$$\liminf_{n\to\infty} \|f_n\|_X \le \infty.$$

Then $f \in X(\Omega)$ and

$$\|f\|_X \le \liminf_{n\to\infty} \|f_n\|_X.$$

Minkowski's integral inequality for function norms $\|\|f(x,y)\|_Y\|_X \le M\|\|f(x,y)\|_X\|_Y$ holds for all measurable functions and some fixed constant M whenever there is $p \in [1,\infty]$ such that $\|\cdot\|_X$ is p-concave and $\|\cdot\|_Y$ is p-convex [22]. In particular,

$$\left\| \int_A f(\cdot,y)\,dy \right\|_X \le \int_A \|f(\cdot,y)\|_X\,dy. \tag{2.1}$$

Also, we briefly provide some notions and facts from the analysis in Banach spaces. Let V be a Banach space. A function $u : \Omega \to V$ is said to be *(strongly) measurable* if there is a sequence of simple functions $u_k = \sum_{i=1}^{N_k} v_i \chi_{A_i}$, $v_i \in V$ such that $\|u - u_k\|_V \to 0$ a.e. on Ω. And function $u : \Omega \to V$ is *weakly measurable* if $\langle v^*, u\rangle$ is measurable for all $v^* \in V^*$. We say that u is *almost separably valued* if there exists a set Σ of measure zero such that $u(\Omega \setminus \Sigma)$ is separable. The strong and weak measurability are compared in the next theorem (see also [19, Theorem 1.1], [14, Theorem 1.1.20]).

Theorem 2.2 (*Pettis Measurability Theorem*). *A function $u : \Omega \to V$ is measurable if and only if it is weakly measurable and almost separably valued.*

There is the theory of Bochner integral, which allows us to integrate vector-valued functions and supplies us with all necessary tools. By $X(\Omega; V)$ we denote the collection of all strongly measurable functions $u : \Omega \to V$ for which $\|u(\cdot)\|_V \in X(\Omega)$. Together with the norm $\|u\|_{X(\Omega;V)} = \big\|\|u(\cdot)\|_V\big\|_{X(\Omega)}$, it becomes a Banach space (see [16, p. 177]). We say that $\tilde{u}$ is *a representative* of u if $u = \tilde{u}$ a.e.

There are several notions connected to absolute continuity that we use. A function $u : [a,b] \to V$ is said to be *absolutely continuous* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^m \|u(b_i) - u(a_i)\|_V \le \varepsilon$ for any collection of disjoint intervals $\{[a_i,b_i]\} \subset [a,b]$ such that $\sum_{i=1}^m (b_i - a_i) \le \delta$. A function $u : \Omega \to V$ is said to be *absolutely continuous on a curve* γ in Ω if $\gamma : [0, l(\gamma)] \to \Omega$ is rectifiable, parametrized by the arc length, and the function $u \circ \gamma : [0, l(\gamma)] \to V$ is absolutely continuous. A function $u : \Omega \to V$ is said to be *absolutely continuous on lines* in Ω (belongs to $ACL(\Omega)$) if u is absolutely continuous on almost every compact line segment in Ω parallel to the coordinate axes.

Let (Ω, Σ, μ) be a σ-finite complete measure space. A Banach space V has the *Radon–Nikodým property* (RNP) if for any measure $\nu : \Sigma \to V$ with bounded variation that is absolutely continuous with respect to μ, there exists a function $f \in L^1(\Omega; V)$ such that $\nu(A) = \int_A f\,d\mu$ for all $A \in \Sigma$. However, for our purposes we make use of equivalent descriptions for this property:

Proposition 2.3 *([14, Theorem 2.5.12]). For any Banach space V, the following assertions are equivalent:*

(1) V has the Radon–Nikodým property;
(2) every locally absolutely continuous function $f : \mathbb{R} \to V$ is differentiable almost everywhere;
(3) every locally Lipschitz continuous function $f : \mathbb{R} \to V$ is differentiable almost everywhere.

Note that each reflexive space has the RNP, and so does every separable dual (V is *separable dual* if it is separable and there is a Banach space Y such that $V = Y^*$). On the other hand, there are spaces that do not have the RNP, such as ℓ^∞, c_0, $L^1([0,1])$. For more information on the RNP, see in [2, Chapter 5].

3. Sobolev spaces based on Banach function spaces

A function $v \in L^1_{loc}(\Omega; V)$ is said to be a weak partial derivative with respect to jth coordinate of the function $u \in L^1_{loc}(\Omega; V)$ if

$$\int_\Omega \frac{\partial \varphi}{\partial x_j}(x) u(x)\, dx = -\int_\Omega \varphi(x) v(x)\, dx$$

for all $\varphi \in C_0^\infty(\Omega)$. In this case we denote $v = \partial_j u$. The Sobolev space $W^1X(\Omega; V)$ is the space of all $u \in X(\Omega; V)$ whose weak derivatives exist and belong to $X(\Omega; V)$. On $W^1X(\Omega; V)$ we define a norm

$$\|u\|_{W^1X} = \|u\|_{X(\Omega;V)} + \||\nabla u|\|_{X(\Omega)},$$

where $|\nabla u| = \sqrt{\sum_{j=1}^n \|\partial_j u\|_V^2}$. In the case of real-valued functions we will use $W^1X(\Omega)$ instead of $W^1X(\Omega; \mathbb{R})$.

If the norm $\|\cdot\|_X$ is absolutely continuous and has the translation inequality property, then the Meyers–Serrin theorem holds true: $C^\infty(\Omega; V) \cap W^1X(\Omega; V)$ is dense in $W^1X(\Omega; V)$ with respect to the norm $\|\cdot\|_{W^1X}$. In this case, Sobolev functions are approximated with the help of standard mollification technique [8, Corollary 3.1.5].

The Sobolev X-capacity of a set $E \subset \Omega$ is defined as

$$\mathrm{Cap}_X(E) = \inf\{\|u\|_{W^1X} : u \geq 1 \text{ on } E\}.$$

Theorem 3.1. *Let $u_i \in C^\infty(\Omega) \cap W^1X(\Omega)$ and $\{u_i\}$ is a Cauchy sequence in $W^1X(\Omega)$. Then there is a subsequence of $\{u_i\}$ that converges pointwise in Ω except a set of X-capacity zero. Moreover, the convergence is uniform outside a set of arbitrarily small X-capacity.*

Proof. This can be proved analogously to the L^p case. □

Theorem 3.2. *If $u \in W^1X(\Omega)$, then there is a representative $\tilde{u}$, which is absolutely continuous and differentiable almost everywhere on lines in Ω. Moreover, $\frac{\partial \tilde{u}}{\partial x_j} = \partial_j u$ a.e.*

Proof. This follows from the fact that $W^1X(\Omega) \subset W^{1,1}_{loc}(\Omega)$. □

3.1. Reshetnyak–Sobolev space

The ultimate aim of this subsection is to provide condition on space V under which functions from $R^1X(\Omega; V)$ possess weak derivatives. We develop the ideas that we learned from [10, Section 2] by P. Hajłasz and J. Tyson. The key modification is that we change the assumption "V is dual to separable" to "V has the RNP".

The *Reshetnyak–Sobolev space* $R^1X(\Omega;V)$ is the class of all functions $u \in X(\Omega;V)$ such that:

(A) for every $v^* \in V^*$, $\|v^*\| \le 1$, we have $\langle v^*, u\rangle \in W^1X(\Omega)$;
(B) there is a non-negative function $g \in X(\Omega)$ such that

$$|\nabla\langle v^*, u\rangle| \le g \quad \text{a.e. on } \Omega \tag{3.1}$$

for every $v^* \in V^*$ with $\|v^*\| \le 1$.

A function g satisfying condition (B) above is called a *Reshetnyak upper gradient* of u. The norm in $R^1X(\Omega;V)$ is defined via

$$\|u\|_{R^1X} = \|u\|_{X(\Omega;V)} + \inf \|g\|_{X(\Omega)},$$

where the infimum is taken over all Reshetnyak upper gradients of u.

The form of the definition above is given by Yu. G. Reshetnyak ([21, p. 573] for functions valued in a metric space); for functions valued in a Banach space, we refer to [12] and [10].

In the next lemma, which is a modification of [10, Lemma 2.12], we provide sufficient conditions for function u to be in $W^1X(\Omega;V)$.

Lemma 3.3. *Let V be a Banach space enjoying the Radon–Nikodým property. Suppose function $u \in X(\Omega;V)$ is so that for every $j \in \{1,\dots,n\}$ it has a representative $\tilde{u}$, which is absolutely continuous on almost every compact line segment in Ω parallel to x_j-axis and partial derivatives exist and satisfy $\left\|\frac{\partial\tilde{u}}{\partial x_j}\right\|_V \le g$ a.e. for some $g \in X(\Omega)$. Then $u \in W^1X(\Omega;V)$ and $\|u\|_{W^1X} \le \|u\|_{X(\Omega;V)} + \sqrt{n}\|g\|_{X(\Omega)}$.*

Proof. Fix $j \in \{1,\dots,n\}$. Due to the RNP, partial derivative $\frac{\partial\tilde{u}}{\partial x_j}$ exists on almost every compact line segment in Ω parallel to the coordinate axes (Proposition 2.3). Let Γ be a collection of all segments in Ω parallel to the x_j-axis, on which function $\tilde{u}$ fails to be absolutely continuous. Denote $\Sigma = P_j\Gamma$, which is the projection of Γ on subspace orthogonal to the x_j-axis, then $\mu^{n-1}(\Sigma) = 0$. Now, with the help of the Fubini theorem, for any $\varphi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}\int_\Omega u\frac{\partial\varphi}{\partial x_j}\,dx &= \int_\Omega \tilde{u}\frac{\partial\varphi}{\partial x_j}\,dx = \int_{P_j\Omega}\int_{l_j(y)\cap\Omega}\tilde{u}\frac{\partial\varphi}{\partial x_j}\,ds\,dy\\ &= \int_{P_j\Omega\setminus\Sigma}\int_{l_j(y)\cap\Omega}\tilde{u}\frac{\partial\varphi}{\partial x_j}\,ds\,dy = \int_{P_j\Omega\setminus\Sigma}\int_{l_j(y)\cap\Omega}\frac{\partial\tilde{u}}{\partial x_j}\varphi\,ds\,dy = \int_\Omega\frac{\partial\tilde{u}}{\partial x_j}\varphi\,dx,\end{aligned}$$

where $l_j(y)$ is a line parallel to the x_j-axis and passing through $y \in P_j\Omega$. Therefore, u has weak partial derivatives which are in $X(\Omega;V)$. □

Remark 3.4. There are some issues with original lemma 2.12 of [10]. Namely, derivatives that are constructed in its proof are not always strongly measurable. Authors of [10] assume that V is dual to some separable space. However, it seems to be not enough for their purpose. This obstacle had been first noted in [5] and has recently been resolved in [6].

Lemma 3.5. *Let $u \in R^1X(\Omega;V)$. Then for each $j \in \{1,\dots,n\}$ there is a representative $\tilde{u}$ which is absolutely continuous on almost every compact line segment in Ω parallel to x_j-axis. Moreover, the following limit exists and satisfies*

$$\lim_{h\to 0}\frac{\|\tilde{u}(x+he_j)-\tilde{u}(x)\|_V}{h} \le g(x) \quad \text{for a.e. } x \in \Omega, \tag{3.2}$$

where $g \in X(\Omega)$ is a Reshetnyak upper gradient of u.

Proof. The function $u \in R^1X(\Omega;V)$ is measurable; therefore, by the Pettis Theorem 2.2, it is essentially separable valued. In other words, there is a subset $\Sigma_0 \subset \Omega$ of measure zero so that $u(\Omega \setminus \Sigma_0)$ is separable in V. Let $\{v_i\}_{i\in\mathbb{N}}$ be a dense subset in the difference set

$$f(\Omega \setminus \Sigma_0) - f(\Omega \setminus \Sigma_0) = \{f(x) - f(y) \colon x, y \in \Omega \setminus \Sigma_0\},$$

and let $v_i^* \in V^*$, $\|v_i^*\| = 1$, be such that $\|v_i\| = \langle v_i^*, v_i\rangle$ (the last is due to the Hahn–Banach theorem, see [13, p. 17]).

For each $i \in \mathbb{N}$ there is a representative $u_i \in ACL(\Omega)$ of $\langle v_i^*, u\rangle \in W^1X(\Omega)$ (Theorem 3.2), and the inequality $|\nabla u_i| \le g$ holds true. Let $\Sigma_i \subset \Omega$ be a set of measure zero, where u_i differs from $\langle v_i^*, u\rangle$.

Fix $j \in \{1, \dots, n\}$. Then for almost all compact line segments $l : [a, b] \to \Omega$ of the form $l(\tau) = x_0 + \tau e_j$ we have:

(a) g is integrable on l;
(b) $\mu^1(l \cap \Sigma) = 0$, where $\Sigma = \Sigma_0 \cup \bigcup_i \Sigma_i$;
(c) For each $i \in \mathbb{N}$ and every $a \le s \le t \le b$

$$|u_i(x_0 + te_j) - u_i(x_0 + se_j)| \le \int_s^t g(x_0 + \tau e_j)\, d\tau. \tag{3.3}$$

The Fubini theorem ensures (a) and (b), while (c) follows from the estimate $|\nabla u_i| \le g$. Let l be a segment so that (a)–(c) hold true. If $x_0 + se_j \notin \Sigma$ and $x_0 + te_j \notin \Sigma$, then there is a sequence v_{i_k} converging to $u(x_0 + te_j) - u(x_0 + se_j)$ in V. It can be shown that in this case

$$\|u(x_0 + te_j) - u(x_0 + se_j)\|_V \le \limsup_{k\to\infty} |u_{i_k}(x_0 + te_j) - u_{i_k}(x_0 + se_j)|.$$

The last estimate together with (3.3) give us

$$\|u(x_0 + te_j) - u(x_0 + se_j)\|_V \le \int_s^t g(x_0 + \tau e_j)\, d\tau. \tag{3.4}$$

If any of endpoints are in Σ, say $x_0 + se_j \in \Sigma$, then we can choose a sequence $s_k \to s$ so that $x_0 + s_k e_j \in l \setminus \Sigma$. With the help of (3.4), it is easy to see that $u(x_0 + s_k e_j)$ converges in V, and the limit does not depend on the choice of sequence. This allows us to define the desired representative $\tilde{u}(x) = u(x)$ if $x \in \Omega \setminus \Sigma$; $\tilde{u}(x) = \lim_{s_k\to 0} u(x + s_k e_j)$ if there is a segment with x as its endpoint; and we put $\tilde{u}(x) = 0$ in other cases. It is easy to see that (3.4) holds true for $\tilde{u}$, and almost every compact line segment in Ω parallel to x_j-axis. Estimate (3.2) follows immediately. □

We should note that in the lemma above the constructed representatives $\tilde{u}$ does not necessarily belong to $ACL(\Omega)$, but this does not affect our results. However, it is possible to prove stronger property: there is a representative that is absolutely continuous on almost every rectifiable curve γ in Ω, see [5, Theorem 4.5] and the proof of [13, Theorem 7.1.20].

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^n$ be open.*

(1) If $u \in W^1X(\Omega;V)$, then $u \in R^1X(\Omega;V)$. Moreover, $|\nabla u|$ is a Reshetnyak upper gradient of u and $\|u\|_{R^1X} \le \|u\|_{W^1X}$.
(2) If V has the Radon–Nikodým property and $u \in R^1X(\Omega;V)$, then $u \in W^1X(\Omega;V)$ and $\|u\|_{W^1X} \le \sqrt{n}\|u\|_{R^1X}$.

Proof. (1) Let $u \in W^1X(\Omega; V)$, and $v^* \in V^*$ with $\|v^*\| \leq 1$. Then $\langle v^*, u\rangle \in X(\Omega)$ since $\langle v^*, u\rangle$ is measurable (Theorem 2.2) and $|\langle v^*, u\rangle| \leq \|u\|_V$. Using the property of the Bochner integral that $\int \langle v^*, u\rangle = \langle v^*, \int u\rangle$, it is easy to show that $\langle v^*, u\rangle$ has weak derivatives in $X(\Omega)$, and $\partial_j \langle v^*, u\rangle = \langle v^*, \partial_j u\rangle$. Moreover, $|\partial_j \langle v^*, u\rangle| \leq \|\partial_j u\|_V \leq |\nabla u|$ a.e. on Ω.

(2) Let $u \in R^1X(\Omega; V)$. Then Lemma 3.5 and the RNP of V imply the assumptions of Lemma 3.3. Thus, $u \in W^1X(\Omega; V)$. □

Theorem 3.7. *A Banach space V has the Radon–Nikodým property if and only if $R^1X(\Omega; V) = W^1X(\Omega; V)$.*

Proof. Due to Theorem 3.6, it remains to prove that V has the RNP in the case $R^1X(\Omega; V) \subset W^1X(\Omega; V)$. Let $f : I \to V$ be Lipschitz continuous, where I is a bounded interval. We may assume that $\Omega = I$ since we can embed I^d into Ω and treat the function $x \mapsto f(x_1)$. For any $v^* \in V^*$ with $\|v^*\| \leq 1$ function $\langle v^*, f\rangle : I \to \mathbb{R}$ is Lipschitz continuous with the same Lipschitz constant L and its derivative $|\langle v^*, f\rangle'| \leq L$. As constant function $x \mapsto L$ belongs to $X(I)$, by Lemma 3.3 $\langle v^*, f\rangle \in W^1X(I)$. Thus, conditions (A) and (B) are fulfilled; therefore, $f \in R^1X(I; V)$, by the assumption $f \in W^1X(I; V) \subset W^{1,1}_{loc}(I; V)$. From the last fact, we obtain that the derivative f' exist almost everywhere on I. □

Remark 3.8. Theorem 4.6 of [5] exhibits the same phenomenon in the case $X = L^p$, $1 \leq p < \infty$.

There are other definitions of Reshetnyak–Sobolev space.

Theorem 3.9. *Let Ω be a bounded domain and $u : \Omega \to V$ be a measurable function. Then the following four conditions are equivalent:*

(i) $u \in R^1X(\Omega; V)$.
(ii) There exists a non-negative function $\rho \in X(\Omega)$ with the following property: for each 1-Lipschitz function $\varphi : V \to \mathbb{R}$ function $\varphi \circ u \in W^1X(\Omega)$ and $|\nabla \varphi \circ u| \leq \rho$ a.e. on Ω.
(iii) There exists a non-negative function $\rho \in X(\Omega)$ with the following property: for each $v \in u(\Omega)$ function $u_v(x) = \|u(x) - v\|_V$ belongs to $W^1X(\Omega)$ and $|\nabla u_v| \leq \rho$ a.e. on Ω.

Proof. (i)⇒(ii) follows from Theorem 4.2. (ii)⇒(iii) is obvious. To prove (iii)⇒(i), we use the same approach as in the proof of Lemma 3.5. For now we take a dense set $\{v_i\}_{i\in\mathbb{N}}$ in $u(\Omega \setminus \Sigma_0)$. Let $\Sigma_i \subset \Omega$ be a set of measure zero, where $\|u - v_i\|_V$ differs from its absolutely continuous representative. Fix $j \in \{1, \dots, n\}$. Let $l(\tau) = x_0 + \tau e_j$ be a segment in Ω so that (a)–(c) hold true. Then choosing a sequence $v_{i_k} \to u(x_0 + te_j)$ we obtain

$$
\begin{aligned}
|\langle v^*, u(x_0 + te_j)\rangle - \langle v^*, u(x_0 + se_j)\rangle| &\leq \|u(x_0 + te_j) - u(x_0 + se_j)\| \\
&= \lim_{k\to\infty} \big|\|u(x_0 + te_j) - v_{i_k}\|_V - \|u(x_0 + se_j) - v_{i_k}\|_V\big| \leq \int_s^t \rho(x_0 + \tau e_j)\, d\tau.
\end{aligned}
$$

So there is a representative u_{v^*} of $\langle v^*, u\rangle$, which is absolutely continuous on almost every compact line segment in Ω parallel to x_j-axis, and its partial derivative exists and satisfies $\left|\frac{\partial u_{v^*}}{\partial x_j}\right| \leq \rho$. Due to Lemma 3.3 $\langle v^*, u\rangle \in W^1X(\Omega)$, and by the estimate above $|\nabla\langle v^*, u\rangle| \leq \sqrt{n}\rho$. Thus, conditions (A) and (B) are realized. □

3.2. Newtonian space

The concept of Newtonian spaces is based on the Newton–Leibniz formula and employs the idea of estimating the difference of function values in two distinct points by the integral over a curve that connects those points. An extensive study of Newtonian spaces $N^{1,p}$ could be found in [13], whereas, in [18], L. Malý constructed the theory of Newtonian spaces based on quasi-Banach function lattices. Here we make use of elements of the theory from [18], taking into account that $X(\Omega)$, in particular, is a quasi-Banach function lattice.

The X-*modulus* of the family of curves Γ is defined by

$$\operatorname{Mod}_X(\Gamma) = \inf \|\rho\|_{X(\Omega)},$$

where the infimum is taken over all non-negative Borel functions ρ that satisfy $\int_\gamma \rho\, ds \geq 1$ for all $\gamma \in \Gamma$ (such functions are called admissible densities for Γ).

Lemma 3.10 (*Estimates for Cylindrical Curve Families*)**.** *Consider a cylinder $G = E \times J$, where E is a Borel set in $\mathbb{R}^{n-1}$ with $\mu^{n-1}(E) < \infty$, and $J \subset \mathbb{R}$ is an interval of length $h \in (0, \infty)$. Let $\Gamma(E)$ be the family of all curves $\gamma_y : J \to G$, $\gamma_y(t) = (y, t)$ for $y \in E \setminus \Sigma$, with $\mu^{n-1}(\Sigma) = 0$. Then*

$$\mu^{n-1}(E) \leq \|\chi_G\|_{X'} \cdot \operatorname{Mod}_X(\Gamma(E)) \tag{3.5}$$

and

$$\operatorname{Mod}_X(\Gamma(E)) \leq \|\chi_G\|_X \cdot h^{-1}. \tag{3.6}$$

Proof. Let ρ be an admissible density for $\Gamma(E)$. By the Fubini theorem and Hölder's inequality we have

$$\mu^{n-1}(E) \leq \int_E \int_{\gamma_y} \rho\, ds\, dy = \int_G \rho\, dx \leq \|\rho\|_X \cdot \|\chi_G\|_{X'},$$

which implies (3.5). To obtain (3.6), we observe that $\frac{1}{h} \cdot \chi_G$ is an admissible density for $\Gamma(E)$. $\square$

The next lemma is a modification of [5, Lemma 2.4]

Lemma 3.11. *Let H be a hyperplane in $\mathbb{R}^n$ and $P_H : \mathbb{R}^n \to H$ be the orthogonal projector. Suppose we are given some family Γ consisting of line segments orthogonal to H. If $\operatorname{Mod}_X(\Gamma) = 0$, then $\mu^{n-1}(P_H\Gamma) = 0$.*

Proof. Let $w \in R^n$ be a unit normal of H. Each curve in Γ is of the form $\gamma_y = y + wt$, for some $y \in H$, and defined on some interval $a \leq t \leq b$.

To prove the assertion we construct a countable family of sets of measure zero which form a covering of $P_H\Gamma$. For $k \in \mathbb{N}$, we consider a $(n-1)$-ball $B(y_0, k)$ in H with centre y_0 and radius k. Then, we pick a subfamily Γ_k in the following way: for each $y \in B_k^{n-1}$ take one if any $\gamma \in \Gamma$ so that $P_H\gamma = y$ and γ is defined on interval $[a, b] \subset [-k, k]$ (see Fig. 1).

Denote $E_k := P_H\Gamma_k$ and take a Borel set $\tilde{E}_k \supset E_k$ with the property $\mu^{n-1}(\tilde{E}_k) = \mu^{n-1}(E_k)$. Consider an additional family $\tilde{\Gamma}_k$ consisting of curves $\gamma_y(t) = y + wt$ for each $y \in \tilde{E}_k$ defined on interval $[-k, k]$. Making use subadditivity of Mod_X and estimate (3.6), we conclude that $\operatorname{Mod}_X(\tilde{\Gamma}_k) = 0$. Then, $\tilde{\Gamma}_k$ and $\tilde{E}_k$ form a cylinder G_k with base $\tilde{E}_k$ and height $2k$.

Therefore, due to estimate (3.5)

$$\mu^{n-1}(E_k) = \mu^{n-1}(\tilde{E}_k) \leq \|\chi_{G_k}\|_{X'} \cdot \operatorname{Mod}_X(\tilde{\Gamma}_k) = 0.$$

So

$$\mu^{n-1}(P_H\Gamma) \leq \sum_k \mu^{n-1}(E_k) = 0. \quad \square$$

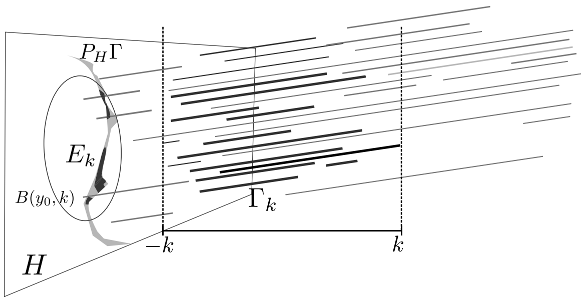

Fig. 1. Subfamily Γ_k and its projection E_k on hyperplane H. The grey area is $P_H\Gamma_k$, and the black area is the projection $P_H\Gamma_k$.

Lemma 3.12 (*Fuglede's Lemma*)**.** *Assume that $g_k \to g$ in $X(\Omega)$ as $k \to \infty$. Then, there is a subsequence (which we still denote by $\{g_k\}$) such that*

$$\int_\gamma g_k\, ds \to \int_\gamma g\, ds, \qquad \text{as } k \to \infty$$

for Mod_X*-a.e. curve γ, while all the integrals are well defined and real-valued.*

Lemma 3.13 (*[17, Proposition 5.10.]*)**.** *Let $E \subset \Omega$ be an arbitrary set, define $\Gamma_E = \{\gamma \in \Gamma(\Omega) \colon \gamma^{-1}(E) \neq \emptyset\}$, which is the collection of all curves in Ω that meet E. If $\operatorname{Cap}_X(E) = 0$, then $\operatorname{Mod}_X(\Gamma_E) = 0$.*

The Newtonian space $N^1X(\Omega;V)$ consists of all functions $u \in X(\Omega;V)$ for which there is a non-negative Borel function $\rho \in X(\Omega)$ such that

$$\|u(\gamma(0)) - u(\gamma(l_\gamma))\|_V \leq \int_\gamma \rho\, ds$$

for Mod_X-a.e. curve γ in Ω. Each such function ρ is called X-*weak upper gradient* of u. Define a semi-norm on $N^1X(\Omega;V)$ via

$$\|f\|_{N^1X} = \|f\|_{X(\Omega;V)} + \inf \|\rho\|_{X(\Omega)},$$

where the infimum is over all X-weak upper gradients of u. Furthermore, we assume that $N^1X(\Omega;V)$ consists of equivalence classes of functions, where $u_1 \sim u_2$ means $\|u_1 - u_2\|_{N^1X} = 0$. We write $N^1X(\Omega)$ instead of $N^1X(\Omega;\mathbb{R})$.

Theorem 3.14. *Let Ω be a domain in $\mathbb{R}^n$ and $X(\Omega)$ be a Banach function space.*

(1) If $u \in N^1X(\Omega)$, then $u \in W^1X(\Omega)$ and $|\nabla u| \leq \sqrt{n}\rho$ a.e. on Ω, where ρ is any X-weak upper gradient of u.

(2) Suppose norm $\|\cdot\|_X$ is absolutely continuous and has the translation inequality property. If $u \in W^1X(\Omega)$, then there is a representative $\tilde{u} \in N^1X(\Omega)$, and as a X-weak upper gradient of $\tilde{u}$, one can choose a Borel representative of $|\nabla u|$.

Proof. (1) Let $u \in N^1X(\Omega)$ and $\rho \in X(\Omega)$ be a X-weak upper gradient of u. Function u is absolutely continuous on Mod_X-a.e. curve γ in Ω. Thanks to Lemma 3.10, u is absolutely continuous on almost all lines parallel to coordinate axes. Moreover, $\left|\frac{\partial u}{\partial x_j}\right| \leq \rho$ a.e. on such lines. Thus, applying Lemma 3.3, we infer that $u \in W^1X(\Omega)$.

(2) Let $u \in W^1X(\Omega)$, then there is a sequence of smooth functions $\{u_k\}$ so that $u_k \to u$ and $\nabla u_k \to \nabla u$ in $X(\Omega)$, as $k \to \infty$. For any curve γ we have

$$|u_k(\gamma(0)) - u_k(\gamma(l_\gamma))| \leq \int_\gamma |\nabla u_k|\, ds.$$

Choose a Borel representative of $|\nabla u|$, then, by Fuglede's Lemma 3.12

$$\int_\gamma |\nabla u_k|\, ds \to \int_\gamma |\nabla u|\, ds \quad \text{as } k\to\infty$$

holds for Mod_X-a.e. curve. Furthermore, due to Theorem 3.1, we can assume that $u_k \to u$ pointwise, except a set E of capacity zero. On the other hand, by Lemma 3.13, X-modulus of the family of curves that meet E is zero. Therefore, we can pass to the limit and obtain that

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \le \int_\gamma |\nabla u|\, ds$$

holds for Mod_X-almost every curve. □

Theorem 3.15. *Let Ω be a domain in $\mathbb{R}^n$, V be a Banach space, and $X(\Omega)$ be a Banach function space.*

(1) If $u \in N^1X(\Omega;V)$, then $u \in R^1X(\Omega;V)$ and $\sqrt{n}\rho$ is its Reshetnyak upper gradient, where ρ is arbitrary X-weak upper gradient of u.

(2) Suppose the norm $\|\cdot\|_X$ is absolutely continuous and has the translation inequality property. If $u \in R^1X(\Omega;V)$, then there is a representative $\tilde u \in N^1X(\Omega;V)$. Moreover, a Borel representative of any Reshetnyak upper gradient of u is X-weak upper gradient of $\tilde u$.

Proof. (1) Let ρ be a X-weak upper gradient of u. For any $v^* \in V^*$ with $\|v^*\| \le 1$ and curve γ, we have

$$|\langle v^*, u\rangle(\gamma(0)) - \langle v^*, u\rangle(\gamma(l_\gamma))| \le \|u(\gamma(0)) - u(\gamma(l_\gamma))\| \le \int_\gamma \rho\, ds$$

Therefore, $\langle v^*, u\rangle \in N^1X(\Omega)$ with X-weak upper gradient ρ, which is not depend on v^*. Due to Theorem 3.14, $\langle v^*, u\rangle \in W^1X(\Omega)$ and $|\nabla\langle v^*, u\rangle| \le \sqrt{d}\cdot\rho$. So $u \in R^1X(\Omega;V)$.

(2) Let $u \in R^1X(\Omega;V)$ and $g \in X(\Omega)$ be its Reshetnyak upper gradient. Then, due to Theorem 3.14, for any $v^* \in V^*$ with $\|v^*\| \le 1$, function $\langle v^*, u\rangle$ has a representative in $N^1X(\Omega)$. Moreover, a Borel representative of g is a X-weak upper gradient for each of those representatives above (not depending on v^*). Therefore, to construct the desired representative of u, we can proceed as in the proof of Lemma 3.5 (also see the proof from [13, p. 182–183]). □

3.3. Description via difference quotients

Here we extend the characterization of Sobolev spaces via difference quotients known for L^p-spaces to the case of Banach function spaces. For the real-valued case see [3, Theorem 2.1.13] and [4, Proposition 9.3], and for the vector case see [14, Proposition 2.5.7] and [1, Theorem 2.2].

Theorem 3.16. *Let $X(\Omega)$ be a Banach function space having the Radon–Nikodým property. If $u \in X(\Omega)$ and there is a constant $C \in [0,\infty)$ such that*

$$\|\tau_{te_j}u - \tau_{se_j}u\|_{X(\omega)} \le C|t-s|, \quad j \in 1,\dots,n \tag{3.7}$$

for all $\omega \Subset \Omega$ with $\max\{|t|,|s|\} < \operatorname{dist}(\omega,\partial\Omega)$, then $u \in W^1X(\Omega)$ and $\|\nabla u\|_{X(\Omega)} \le nC$.

Hereinafter $\omega \Subset \Omega$ means that the closure of ω is a compact subset of Ω.

Proof. Fix $j \in 1,\dots,n$ and let $\omega \Subset \Omega$ be bounded. First, we prove that weak derivatives of $u|_\omega$ exist in $X(\omega)$ and their norms are bounded by C. Let $\omega \Subset \omega' \Subset \Omega$ and $0 < \delta < \operatorname{dist}(\omega',\partial\Omega)$. Consider function $G : (-\delta,\delta) \to X(\omega')$ defined by the rule $t \mapsto \tau_{te_j}u$. By the assumption (3.7), we have

$$\|G(t) - G(s)\|_{X(\omega')} = \|\tau_{te_j}u - \tau_{se_j}u\|_{X(\omega)} \le C|t-s|,$$

meaning that G is Lipschitz continuous. Due to the RNP of X, mapping G is differentiable a.e. Then fix $0 \le t_0 < \operatorname{dist}(\omega, \partial\omega')$ so that

$$G'(t_0) = \lim_{h\to 0} \frac{u(\cdot + (t_0 + h)e_j) - u(\cdot + t_0 e_j)}{h} \tag{3.8}$$

exists in $X(\omega')$. Choose a sequence $h_k \to 0$ such that limit (3.8) exist a.e. in ω' and, in particular, in $\omega - t_0 e_j \subset \omega'$. For $x \in \omega$, we define

$$g_\omega(x) := \lim_{k\to\infty} \frac{u(x + h_k e_j) - u(x)}{h_k}.$$

Then g_ω is measurable, and by Lemma 2.1, $g_\omega \in X(\omega)$ with $\|g_\omega\|_{X(\omega)} \le C$. Denote $g^k_\omega(x) = \frac{u(x+h_k e_j)-u(x)}{h_k}$ and show that for any $\varphi \in C_0^\infty(\omega)$ the next equality holds

$$\lim_{k\to\infty} \int_\omega g^k_\omega(x)\varphi(x)\,dx = \int_\omega g_\omega(x)\varphi(x)\,dx.$$

Indeed:

$$\begin{aligned}\left|\int_\omega g^k_\omega(x)\varphi(x)\,dx - \int_\omega g_\omega(x)\varphi(x)\,dx\right| &\le \int_\omega |g^k_\omega(x) - g_\omega(x)|\cdot|\varphi(x)|\,dx \\ &= \int_{\omega - t_0 e_j} |g^k_\omega(y + t_0 e_j) - G'(t_0)(y)|\cdot|\varphi(y + t_0 e_j)|\,dy \\ &\le \left\|\frac{u(\cdot + (t_0 + h_k)e_j) - u(\cdot + t_0 e_j)}{h_k} - G'(t_0)\right\|_{X(\omega')} \|\varphi(\cdot + t_0 e_j)\|_{X'(\omega')} \to 0.\end{aligned}$$

We deduce that g_ω is a weak derivative:

$$\begin{aligned}\int_\omega g_\omega(x)\varphi(x)\,dx &= \lim_{k\to\infty}\int_\omega \frac{u(x + h_k e_j) - u(x)}{h_k}\varphi(x)\,dx \\ &= \lim_{k\to\infty}\int_\omega \frac{\varphi(x + h_k e_j) - \varphi(x)}{h_k} u(x)\,dx = \int_\omega u(x)\frac{\partial\varphi}{\partial x_j}(x)\,dx.\end{aligned}$$

Now we take a monotone sequence of bounded domains $\omega_n \Subset \omega_{n+1} \Subset \Omega$ such that $\bigcup_n \omega_n = \Omega$. Functions g_{ω_n} agree on the intersections of their supports; therefore, they can be pieced together to a globally defined measurable function g. Once again, thanks to Lemma 2.1, $g \in X(\Omega)$ and $\|g_\omega\|_{X(\Omega)} \le C$. In the same manner as above, we derive that $g = \partial_j u$ on Ω. □

Theorem 3.17. *Let V be a Banach space and $X(\Omega)$ be a Banach function space. If $u \in X(\Omega; V)$ and there is a constant $C \in [0, \infty)$ such that*

$$\|\tau_{te_j} u - \tau_{se_j} u\|_{X(\omega;V)} \le C|t - s|, \quad j \in 1, \dots, n$$

for all $\omega \Subset \Omega$ with $\max\{|t|, |s|\} < \operatorname{dist}(\omega, \partial\Omega)$, then $u \in R^1 X(\Omega; V)$, and there is g a Reshetnyak upper gradient of u so that $\|g\|_{X(\Omega)} \le nC$.

Proof. For any $v^* \in V^*$ with $\|v^*\| \le 1$, it is clear that $\langle v^*, u\rangle \in X(\Omega)$. Have the following estimate

$$\begin{aligned}|\tau_{te_j}\langle v^*, u\rangle(x) - \tau_{se_j}\langle v^*, u\rangle(x)| &= |\langle v^*, \tau_{te_j}u(x)\rangle - \langle v^*, \tau_{se_j}u(x)\rangle| \\ &= |\langle v^*, \tau_{te_j}u(x) - \tau_{se_j}u(x)\rangle| \le \|\tau_{te_j}u(x) - \tau_{se_j}u(x)\|_V.\end{aligned}$$

Then, for any $\omega \Subset \Omega$ with $\max\{|t|, |s|\} < \operatorname{dist}(\omega, \partial\Omega)$, we have

$$\|\tau_{te_j}\langle v^*, u\rangle - \tau_{se_j}\langle v^*, u\rangle\|_{X(\omega)} \le \|\tau_{te_j}u - \tau_{se_j}u\|_{X(\omega;V)} \le C|t - s|.$$

Thus, all the assumptions of Theorem 3.16 are fulfilled. So $\langle v^*, u\rangle \in W^1X(\Omega)$. Now, find a majorant. Define

$$g_j(x) := \liminf_{h\to 0} \frac{\|u(x+he_j)-u(x)\|_V}{|h|},$$

which belongs to $X(\Omega)$ and $\|g_j\|_{X(\Omega)} \le C$ (due to Lemma 2.1). Applying the next estimate

$$\frac{|\langle v^*, u(x+h)\rangle - \langle v^*, u(x)\rangle|}{|h|} \le \frac{\|u(x+he_j)-u(x)\|_V}{|h|},$$

we derive that

$$\begin{aligned}|\partial_j\langle v^*, u\rangle(x)| &= \lim_{h\to 0}\frac{|\langle v^*, u(x+h)\rangle - \langle v^*, u(x)\rangle|}{|h|}\\ &\le \liminf_{h\to 0}\frac{\|u(x+he_j)-u(x)\|_V}{|h|} = g_j(x).\end{aligned}$$

So $g = \sqrt{\sum_j g_j^2}$ is a Reshetnyak upper gradient of u, and the estimate $\|g\|_{X(\Omega)} \le \sum_j \|g_j\|_{X(\Omega)} \le nC$ holds true. □

Theorem 3.18. *Let V be a Banach space and $X(\Omega)$ be a Banach function space.*

(1) Suppose the norm $\|\cdot\|_X$ is absolutely continuous and has the translation inequality property. If $u \in W^1X(\Omega; V)$, then

$$\|\tau_{te_j}u - \tau_{se_j}u\|_{X(\omega;V)} \le \|\partial_j u\|_{X(\Omega;V)}|t-s|, \quad j \in 1,\dots,n \tag{3.9}$$

for all $\omega \Subset \Omega$ with $\max\{|t|,|s|\} < \operatorname{dist}(\omega,\partial\Omega)$.

(2) Suppose X and V have the Radon–Nikodým property. If $u \in X(\Omega;V)$ and there is a constant $C \in [0,\infty)$ such that

$$\|\tau_{te_j}u - \tau_{se_j}u\|_{X(\omega;V)} \le C|t-s|, \quad j \in 1,\dots,n \tag{3.10}$$

for all $\omega \Subset \Omega$ with $\max\{|t|,|s|\} < \operatorname{dist}(\omega,\partial\Omega)$, then $u \in W^1X(\Omega;V)$ and $\|\nabla u\|_{X(\Omega)} \le nC$.

Proof. (1) By the density it is sufficient to consider $u \in C^\infty(\Omega;V) \cap W^1X(\Omega;V)$. Then,

$$u(x+te_j) - u(x+se_j) = \int_s^t \frac{d}{dr}u(x+re_j)\,dr = \int_s^t \frac{\partial}{\partial x_j}u(x+re_j)\,dr.$$

Applying Minkowski's inequality (2.1) and then the translation inequality property, we derive (3.9).

(2) It is a consequence of Theorems 3.6 and 3.17. □

In [1], W. Arendt and M. Kreuter obtained the following characterization of the Radon–Nikodým property: *A Banach space V has the RNP iff the difference quotient criterion* (3.10) *characterizes the space $W^{1,p}(\Omega;V)$, $p \in (1,\infty]$*. We are interested whether there exists such kind of property for a base space $X(\Omega)$. Namely, we suppose that the following would be reasonable.

Conjecture 3.19. *If the difference quotient criterion* (3.7) *characterizes the space $W^1X(\Omega)$, then the Banach function space $X(\Omega)$ has the Radon–Nikodým property.*

At least for L^p-spaces, it is true.

3.4. A maximal function characterization

Another fruitful observation consists in pointwise description of Sobolev functions via maximal function.

Theorem 3.20. *Let $\Omega \subset \mathbb{R}^n$, V be a Banach space, $X(\Omega)$ have the Radon–Nikodým property, and the norm $\|\cdot\|_{X(\Omega)}$ have the translation inequality property. If $u \in X(\Omega; V)$ and there is a non-negative function $h \in X(\Omega)$ such that*

$$\|u(x) - u(y)\|_V \leq |x - y|(h(x) + h(y)), \qquad \text{a.e. on } \Omega, \tag{3.11}$$

then $u \in R^1X(\Omega; V)$ and $\|g\|_{X(\Omega)} \leq 2n\|h\|_{X(\Omega)}$, where g is some Reshetnyak upper gradient of u.

Proof. For all $j = 1, \dots, n$ and any $\omega \Subset \Omega$ with $\max\{|t|, |s|\} < \operatorname{dist}(\omega, \partial\Omega)$, taking into account the translation inequality property, we deduce

$$\|\tau_{te_j}u - \tau_{se_j}u\|_{X(\omega;V)} \leq |t - s| \cdot \|\tau_{te_j}h + \tau_{se_j}h\|_{X(\omega)} \leq |t - s| \cdot 2\|h\|_{X(\Omega)}.$$

By Theorem 3.17, we conclude that $u \in R^1X(\Omega; V)$. $\square$

Corollary 3.21. *In the assumptions of Theorem 3.20 suppose that V has the Radon–Nikodým property. Then it follows that $u \in W^1X(\Omega; V)$ and $\|\nabla u\|_{X(\Omega)} \leq 2n\|g\|_{X(\Omega)}$.*

A sufficiency counterpart to Theorem 3.20 (and to Corollary 3.21) sounds in the following way:

Theorem 3.22. *Let $\Omega \subset \mathbb{R}^n$, V be a Banach space, $X(\Omega)$ be a Banach function space such that Hardy–Littlewood maximal operator M is bounded in $X(\Omega)$. If $u \in W^1X(\Omega; V)$, then*

$$\|u(x) - u(y)\|_V \leq C|x - y|\big(M(|\nabla u|)(x) + M(|\nabla u|)(y)\big)$$

holds for some constant C and almost all $x, y \in \Omega$ with $B(x, 3|x - y|) \subset \Omega$.

This result has been recently obtained in [15] by P. Jain, A. Molchanova, M. Singh, and S. Vodopyanov for the real-valued case. It is easy to see that the proof of [15, Theorem 2.2] works for vector-valued functions as well.

4. Composition with Lipschitz continuous function

If $u \in N^1X(\Omega; V)$ and $f : V \to Z$ is Lipschitz continuous with $f(0) = 0$, then it is obvious that $f \circ u$ belongs to $N^1X(\Omega; Z)$ and $\operatorname{Lip}(f)\cdot\rho$ is its X-weak upper gradient. Here we discuss superpositions of Lipschitz mapping and functions from classes W^1X and R^1X.

Theorem 4.1. *Suppose that V and Z are Banach spaces such that Z has the Radon–Nikodým property, and $X(\Omega)$ is a Banach function space. Let $f : V \to Z$ be Lipschitz continuous and assume that $f(0) = 0$ if $|\Omega| = \infty$. Then $f \circ u \in W^1X(\Omega; Z)$ for any $u \in W^1X(\Omega; V)$.*

Proof. Let $u \in W^1X(\Omega; V)$. There is a representative $\tilde{u}$ which is absolutely continuous on lines in Ω; then the same holds for $f \circ \tilde{u}$, which is a representative of $f \circ u$. Due to the RNP of Z there exist partial derivatives

$\frac{\partial f\circ\tilde{u}}{\partial x_j}$. For almost all $x\in\Omega$ we have

$$\begin{aligned}\left\|\frac{\partial f\circ\tilde{u}}{\partial x_j}(x)\right\|_Z &= \lim_{h\to 0}\frac{\|f\circ\tilde{u}(x+he_j)-f\circ\tilde{u}(x)\|_Z}{|h|}\\ &\le \lim_{h\to 0} L\frac{\|\tilde{u}(x+he_j)-\tilde{u}(x)\|_V}{|h|} = L\left\|\frac{\partial \tilde{u}}{\partial x_j}(x)\right\|_V = L\|\partial_j u(x)\|_V,\end{aligned}$$

where $L=\mathrm{Lip}(f)$. Let $g(x)=L|\nabla u(x)|$, then, by Lemma 3.3, $f\circ u$ belongs to $W^1X(\Omega;Z)$. □

Theorem 4.2. *Let $\Omega\subset\mathbb{R}^n$ be open, V and Z be Banach spaces, and $f:V\to Z$ be a Lipschitz continuous mapping ($f(0)=0$ in the case $|\Omega|=\infty$). Then, $f\circ u\in R^1X(\Omega;Z)$ whenever $u\in R^1X(\Omega;V)$.*

Proof. Let $u\in R^1X(\Omega;V)$, and $z^*\in Z^*$ with $\|z^*\|\le 1$. It is clear that $f\circ u\in X(\Omega;V)$. Define function $\psi:V\to\mathbb{R}$ by the rule $\psi(v)=\langle z^*,f(v)\rangle$. Then, ψ is Lipschitz continuous, and $\langle z^*,f\circ u\rangle=\psi\circ u$. With the help of Theorem 4.1, the last guarantees $\langle z^*,f\circ u\rangle\in W^1X(\Omega)$ and $|\nabla\langle z^*,f\circ u\rangle|\le Lg$. Thus, we conclude that $f\circ u\in R^1X(\Omega;Z)$. □

Theorem 4.3. *Let $\Omega\subset\mathbb{R}^n$ be open and V, Z be Banach spaces, $V\ne\{0\}$. If for any Lipschitz mapping $f:V\to Z$ we have $f\circ u\in W^1X(\Omega;Z)$ whenever $u\in W^1X(\Omega;V)$, then Z has the Radon–Nikodým property.*

Proof. Suppose Z does not have the RNP. Then there is a Lipschitz function $h:[a,b]\to Z$, which is not differentiable almost everywhere. Fix elements $v_0\in V$ and $v_0^*\in V^*$ so that $\langle v_0^*,v_0\rangle=1$. Consider the next function $f(v)=h(\langle v_0^*,v\rangle)$; it is clear that $f:V\to Z$ is Lipschitz continuous. We can assume that $Q=[a,b]^n\Subset\Omega$. Choose a function $\eta\in C_0^\infty(\Omega)$ such that $\eta(x)=1$ when $x\in Q$. Then, we define function $u(x)=v_0\cdot x_1\eta(x)$, which is in $W^1X(\Omega;V)$. Therefore, by the assumption $f\circ u\in W^1X(\Omega;V)$. On the other hand, $f\circ u(x)=h(x_1)$ when $x\in Q$, meaning that $f\circ u$ is not differentiable almost everywhere on intervals in Q, and this contradicts Theorem 3.2. □

Remark 4.4. We define a nonlinear operator (autonomous Nemytskii operator) $N_f:W^1X(\Omega;V)\to W^1X(\Omega;Z)$ by $N_f u=f\circ u$. Then Theorem 4.1 implies that N_f is bounded. However, the question of whether it is continuous requires additional investigations.

The following lemma is similar to [1, Corollary 3.4. and Corollary 3.4.].

Lemma 4.5. *If $u\in R^1X(\Omega;V)$, then $\|u(\cdot)\|_V\in W^1X(\Omega)$ and $|\partial_j\|u(\cdot)\|_V|\le g$ a.e., where $g\in X(\Omega)$ is a Reshetnyak upper gradient of u.*

Proof. Let $u\in R^1X(\Omega;V)$, then, by Theorem 4.2, $\|u(\cdot)\|_V\in R^1X(\Omega)=W^1X(\Omega)$. Then, with the help of Lemma 3.5, we infer

$$\lim_{h\to 0}\frac{|\|u(x+he_j)\|_V-\|u(x)\|_V|}{|h|}\le\lim_{h\to 0}\frac{\|u(x+he_j)-u(x)\|_V}{|h|}\le g(x)$$

for almost all $x\in\Omega$. □

Theorem 4.6. *Let $\Omega\subset\mathbb{R}^n$ be open such that we have a continuous embedding $W^1X(\Omega)\hookrightarrow Y(\Omega)$ for some Banach function space $Y(\Omega)$. Then we also have a continuous embedding $R^1X(\Omega;V)\hookrightarrow Y(\Omega;V)$.*

Proof. Let $u\in R^1X(\Omega;V)$. Then by Lemma 4.5 $\|u\|_V\in W^1X(\Omega)$, and by the assumption $\|u\|_V\in Y(\Omega)$. The last implies $u\in Y(\Omega;V)$.

Now let C be the norm of the real-valued embedding. Again, using Lemma 4.5, we derive

$$\|u\|_{Y(\Omega;V)} = \big\| \|u\|_V \big\|_{Y(\Omega)} \le C \big\| \|u\|_V \big\|_{W^1X(\Omega)} \le C\sqrt{n}\|u\|_{R^1X(\Omega;V)}. \quad \square$$

As an example we have the classical result of embedding Sobolev space into Lorentz space.

Corollary 4.7. *$W^{1,p}(\mathbb{R}^n;V) \hookrightarrow L^{\frac{pn}{n-p},p}(\mathbb{R}^n;V)$, when $1 \le p < n$.*

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