# SYNCHRONIZATION IN ADAPTIVE KURAMOTO OSCILLATORS FOR POWER GRIDS WITH DYNAMIC VOLTAGES

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Abstract. In this paper we are concerned with a coupled system of Kuramoto oscillators for power grids with dynamic voltages. Among various models describing the dynamics of synchronous generators, analytic results are available mainly for the second-order model with constant voltages which describes only the dynamics of rotor angles. It is known that two of the most important forms for power system stability are rotor angle stability and voltage stability. In [Eur. Phys. J., 223 (2014) 2577-2592], Schmietendorf, Peinke, Friedrich, and Kamps derived a model of adaptive Kuramoto oscillators for the power grids with dynamic voltages and carried out some numerical studies. In this model, the transient dynamics of voltages is incorporated and the voltage dynamics could be considered together with rotor angle dynamics. In this article, we will consider this model and derive some analytic results for the synchronization of phase angles and stabilization of voltages. We will find a region of attraction for a class of steady states which is explicitly expressed in the parameters of system.

## 1. Introduction

Synchronization processes appear in various natural systems such as synchronous firing of swarms of fireflies or clapping audiences, synchronous pacing of heart cells or neurons, synchronization of Josephson junctions arrays [1, 3, 15, 30]. There is a notable relation between power system stability and synchronization phenomena in coupled systems as the synchronous motors' dynamics can be modelled by a modified version of the prominent Kuramoto model (KM) [16]. As a complex and large-scale system, the power grid has rich nonlinear dynamics, and its synchronization and transient stability are very important issues in relation to the power grid safety. It is believed that the future power will rely increasingly on renewables such as wind and solar power which bring an increasing number of transient disturbances. This poses an increasing interest and challenge for us to investigate its synchronous behaviours with large disturbance.

The dynamics of classic Kuramoto model is governed by the following first-order system of all-to-all coupled oscillators:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i),$$

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where $\theta_i(t)$ is the phase of $i$-th oscillator, $K$ and $\Omega_i$ are constants denoting the coupling strength and natural frequency of $i$-th oscillator, respectively. This model has been studied in many literature and a central problem is to find conditions on the parameters and/or initial configurations leading to the existence or emergence of synchronization or phase-locking, see for example, [6, 10, 14, 17, 24, 34]. There are also literature on the Kuramoto model with a network structure, for example, [13, 19, 22, 26]. For the existence of stable synchronized solutions to Kuramoto model on networks, Jafarpour and Bullo [22] used the cutset projection operator to obtain sufficient conditions, which are the sharpest bounds to the best of our knowledge. In [15] Ermentrout introduced the second-order Kuramoto model with inertia effect

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad m > 0$$

to explain the slow synchronization of certain biological systems. The inertia can lead to rich dynamics as noticed in literature, see [7, 9, 32] for example.

It is well-known that the power network systems are closely related to the nonuniform second-order Kuramoto model

$$m_i\ddot{\theta}_i + \gamma_i\dot{\theta}_i = P_{m,i} - \sum_{j=1}^{N} a_{ij}\sin(\theta_i - \theta_j), \tag{1.1}$$

where $\theta_i$ and $\dot{\theta}_i$ are the rotor angle and frequency of the $i$-th generator, respectively. The parameter $P_{m,i}, m_i > 0$ and $\gamma_i > 0$ are the effective power input, inertia constant and damping coefficient of the $i$-th generator, respectively. The coefficient $a_{ij} > 0$ represents the coupling between generators which is given by $a_{ij} = B_{ij}E_iE_j$ where $B_{ij}$ denotes the susceptance of the transmission line between $i$ and $j$ and $E_i$ is the voltage level of $i$th generator. There have been many studies on this model by assuming a constant voltage level, i.e., the coupling strength $a_{ij}$ is constant. For example, some earlier surveys on transient stability can be found in literature [4, 5]. The transient stability, in terms of power grids, is concerned with the system's ability to reach an acceptable synchronism after a major disturbance such as short circuit caused by lightning, large noises in power injections to network, or abrupt changes in environment. Therefore, a fundamental problem for transient stability is to determine whether the post-fault state is located in the region of attraction of synchronism. Recently, this model attracts more and more attention, for example, the relation between the synchronization and network topologies was studied in [13] and the sync basin estimate was considered in [8, 25]. The connection between first- and second-order models was considered by homotopy arguments in [11] and by singular perturbation analysis in [12].

In the above literature, the voltage is set as constant and regarded as a system parameter. However, the power system is a complex dynamical system in which the voltage varies in time. Therefore, it is more realistic to regard the voltage as a state variable rather than a constant parameter. In fact, the stability of power systems can be categorized into several forms and two of the most important ones are rotor angle stability and voltage stability [21, 23, 31]: the former refers to the ability of synchronous machines to remain in synchronism and the second one refers to the ability of a power system to maintain steady voltages at all buses after being subjected to a disturbance. In [29], a model for the power system with converter coupled power generation describing the first-order phase dynamics and voltage dynamics was derived and the stability with contraction analysis was considered. As a future problem, they suggested to consider this type of power system with second-order rotor angle dynamics. To the best of our knowledge, for a power system with second-order phase dynamics, most consideration on system (1.1) focuses on the rotor angle stability by assuming constant voltages, see for example [5, 8, 13, 25]. In [31] Schmietendorf *et. al* introduced an extended model for networks of synchronous machines and they incorporated voltage dynamics into the phase dynamics of power grid. A normalized $N$-machine system [31] can be defined by the following equations:

$$\begin{cases} m_i\ddot{\theta}_i + \gamma_i\dot{\theta}_i = P_{m,i} - \displaystyle\sum_{j=1}^{N} B_{ij}E_iE_j\sin(\theta_i-\theta_j) \\ \alpha_i\dot{E}_i = E_{f,i} - E_i + X_i\displaystyle\sum_{j=1}^{N} B_{ij}E_j\cos(\theta_i-\theta_j), \quad i=1,2,\dots,N. \end{cases} \tag{1.2}$$

Here, $E_i$ is the voltage level of $i$th generator and $E_{f,i} > 0$ is the rotor field voltage. The parameters $\alpha_i$ and $X_i$ are positive constants. For $B = (B_{ij})$ we denote the symmetric nodal admittance matrix satisfying $B_{ij} \le 0$ for $i = j$, and $B_{ij} \ge 0$ for $i \ne j$. The above model is based on a more detailed synchronous machine representation that takes into account the machine's electro-dynamical behaviour to a certain extent. For a detailed derivation of the model we may refer to [28, 31]. Compared to the model (1.1) with constant voltages, the fundamental problem for transient stability of (1.2) is then to determine *whether the post-fault state can evolve towards a phase synchronism and voltage stabilization.* To the best of our knowledge, there is no rigorous study for the dynamical behavior on this model with dynamic voltages. In this paper, we will consider the dynamics of system (1.2) and derive sufficient conditions on parameters and transient state for asymptotic phase synchronism and voltage stabilization.

**Contribution-.** In this paper, we consider two frameworks for the power system. First, we simplify the system (1.2) and consider the first-order dynamics of the phase angles; the reason of this consideration is the connection between the first- and second-order dynamics, which was laid out in [12] for the model with constant voltage. Second, we directly study the system (1.2) with second-order phase dynamics. For these models we derive sufficient conditions leading to asymptotic phase-locking and voltage stabilization, which gives the estimates for the region of attraction of steady states.

We use the gradient system and energy estimate approaches in this paper. For the first-order model, we discuss identical oscillators on a connected graph (Theorem 2.2) and nonidentical oscillators on a network with restricted connectivity that the underlying graph has a diameter $\mathrm{d}(\mathcal{G}) \le 2$ (Theorem 2.3). For the second-order model, we study the network with connected underlying graph (Theorem 2.4) and construct a virtual energy function involving the dynamic voltages to obtain the boundedness of phases and voltages which implies the convergence by gradient approach. To our best knowledge, this is the first analytic study on the coupled dynamics of rotor angles and voltages in power grid with Kuramoto-type model.

**Organization-.** The rest of this paper is organized as follows. In Section 2, we present the models with gradient formulation and our main results. In Section 3, we study the synchronization of first-order system and prove the two main results for identical and nonidentical oscillators respectively. In Section 4, we give a proof for the main result of second-order system. Section 5 is devoted to a concluding summary.

**Notation.**
$\ell^\infty(\mathbb{R}^+, \mathbb{R}^N) = \left\{ f : \mathbb{R}^+ \to \mathbb{R}^N \mid f \text{ is bounded} \right\}, \quad \mathbb{R}^+ := \{t \in \mathbb{R} : t \ge 0\},$
$\ell^{1,\infty}(\mathbb{R}^+, \mathbb{R}^N) = \left\{ f : \mathbb{R}^+ \to \mathbb{R}^N \mid f \text{ is differentiable}, f, f' \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^N) \right\}.$

## 2. Preliminaries

In this section, we present the model of power grids and some elementary estimates. The main results for synchronization are also presented. Let us recall (1.2) and denote $\Omega_i = P_{m,i}, q_i = E_{f,i}$. Since $\sin(\theta_i - \theta_i) = 0$ and $\cos(\theta_i - \theta_i) = 1$, (1.2) can be rewritten as

$$\begin{cases} m_i \ddot{\theta}_i + \gamma_i \dot{\theta}_i = \Omega_i - \displaystyle\sum_{j=1}^N \tilde{B}_{ij} E_i E_j \sin(\theta_i - \theta_j) \\ \alpha_i \dot{E}_i = q_i - \beta_i E_i + X_i \displaystyle\sum_{j=1}^N \tilde{B}_{ij} E_j \cos(\theta_i - \theta_j), \end{cases} \tag{2.1}$$

where

$$\beta_i := 1 - X_i B_{ii} \ge 1, \quad q_i > 0, \quad \tilde{B}_{ij} = \begin{cases} 0, & i = j; \\ \ge 0, & i \ne j. \end{cases}$$

We introduce micro-variables

$$\tilde{\theta}_i = \theta_i - \Omega_s t, \quad \text{with} \quad \Omega_s := \frac{\sum_{i=1}^N \Omega_i}{\sum_{i=1}^N \gamma_i},$$

then we find $\ddot{\tilde{\theta}}_i = \ddot{\theta}_i$, $\dot{\tilde{\theta}}_i = \dot{\theta}_i - \Omega_s$, and

$$\begin{cases} m_i \ddot{\tilde{\theta}}_i + \gamma_i \dot{\tilde{\theta}}_i = \tilde{\Omega}_i - \displaystyle\sum_{j=1}^N \tilde{B}_{ij} E_i E_j \sin(\tilde{\theta}_i - \tilde{\theta}_j) \\ \alpha_i \dot{E}_i = q_i - \beta_i E_i + X_i \displaystyle\sum_{j=1}^N \tilde{B}_{ij} E_j \cos(\tilde{\theta}_i - \tilde{\theta}_j), \end{cases} \tag{2.2}$$

where $\tilde{\Omega}_i := \Omega_i - \gamma_i \Omega_s$ and the micro natural frequencies sum to zero: $\sum_{i=1}^N \tilde{\Omega}_i = 0$. In particular, if $\frac{\Omega_i}{\gamma_i} = \frac{\Omega_j}{\gamma_j}$ for all $i, j = 1, 2, \ldots, N$, then we have $\tilde{\Omega}_i = 0$ for each $i$, that is, it is a system of coupled oscillators with identical natural frequencies. For simplification, we remove the tilde in (2.2) and we obtain the model

$$\begin{cases} m_i \ddot{\theta}_i + \gamma_i \dot{\theta}_i = \Omega_i - \displaystyle\sum_{j=1}^N B_{ij} E_i E_j \sin(\theta_i - \theta_j), \\ \alpha_i \dot{E}_i = q_i - \beta_i E_i + X_i \displaystyle\sum_{j=1}^N B_{ij} E_j \cos(\theta_i - \theta_j), \end{cases} \tag{2.3}$$

subject to initial data

$$\theta_i(0) = \theta_i^0, \quad \omega_i(0) := \dot{\theta}_i(0) = \omega_i^0, \quad E_i(0) = E_i^0,$$

and parameters

$$\sum_{i=1}^N \Omega_i = 0,\ \alpha_i > 0,\ q_i > 0,\ \beta_i \geq 1,\ X_i > 0,\ B_{ij} = \begin{cases} 0, & i = j; \\ \geq 0, & i \neq j. \end{cases}$$

Throughout this paper, we will consider this model instead of (2.1). We next present the definitions of a few concepts for synchronization.

**Definition 2.1.** *Let* $(\theta, E) = (\theta_1, \theta_2, \dots, \theta_N, E_1, E_2, \dots, E_N)^{\mathrm{T}}$ *be a dynamical solution to the system* (2.3)*.*

*(1) The dynamical solution asymptotically exhibits* phase synchronization *if*

$$\lim_{t\to\infty} (\theta_i(t) - \theta_j(t)) = 0, \quad \forall\, i \neq j.$$

*(2) The dynamical solution asymptotically exhibits* frequency synchronization *if*

$$\lim_{t\to\infty} (\dot{\theta}_i(t) - \dot{\theta}_j(t)) = 0, \quad \forall\, i \neq j.$$

*(3) The dynamical solution asymptotically exhibits* phase-locking *if for any pair* $(i, j)$ *there exists a constant* $\theta_{ij}$ *such that*

$$\lim_{t\to\infty} (\theta_i(t) - \theta_j(t)) = \theta_{ij}.$$

*(4) The dynamical solution asymptotically exhibits* voltage stabilization *if there exist* $E_i^\infty$*'s such that*

$$\lim_{t\to\infty} E_i(t) = E_i^\infty, \quad \text{and} \quad \lim_{t\to\infty} \dot{E}_i(t) = 0, \quad i = 1, 2, \dots, N.$$

In this paper, we will consider the synchronization problem in two categories. The first one is to consider a simplified first-order dynamics of the phase angles for which the relevance relies on the singular perturbation analysis. In [12], by singular perturbation analysis the authors showed that the second-order dynamics can be approximated by first-order dynamics. Therefore, we believe that studying the first-order dynamics of the phase angles in (2.3) is of interest. The second one is to analyze the second-order phase dynamics directly involving the voltage dynamics.

2.1. **Models.** The singular perturbation theory by Tikhonov [33] gives an insight to understand the dynamical systems with different time scales as a fast-slow system. Note that the dynamical equations in system (2.3) can be written as the first-order dynamics of $(\theta_i, \omega_i, E_i)$:

$$\dot{\theta}_i = \omega_i, \qquad m_i\dot{\omega}_i = -\gamma_i\omega_i + \Omega_i - \sum_{j=1}^N B_{ij}E_iE_j\sin(\theta_i - \theta_j),$$

$$\alpha_i\dot{E}_i = q_i - \beta_iE_i + X_i\sum_{j=1}^N B_{ij}E_j\cos(\theta_j - \theta_i),$$

When $m_i$ is sufficiently small, we can regard the dynamics of $(\theta_i, E_i)$ as a slow part and $\omega_i$ as a fast part. In [12], this idea was exploited to build a connection between the first- and second-order dynamics of phases. Therefore, we believe that it makes sense to consider the first-order dynamics of phase angles. As a simple situation, we consider the first-order dynamics with $X_i = \gamma_i = \alpha_i = 1$, that is,

$$\text{Model A}: \quad \begin{cases} \dot{\theta}_i = \Omega_i - \sum_{j=1}^{N} B_{ij} E_i E_j \sin(\theta_i - \theta_j), \\ \dot{E}_i = q_i - \beta_i E_i + \sum_{j=1}^{N} B_{ij} E_j \cos(\theta_i - \theta_j), \end{cases} \tag{2.4}$$

subject to

$$\theta_i(0) = \theta_i^0,\ E_i(0) = E_i^0,\ \sum_{i=1}^{N} \Omega_i = 0,\ \beta_i \geq 1,\ q_i > 0,\ B_{ij} = \begin{cases} 0, & i = j; \\ \geq 0, & i \neq j. \end{cases} \tag{2.5}$$

Thanks to the symmetry of coupling strength ($B_{ij}E_iE_j = B_{ji}E_jE_i$) in the dynamics of phases, we have

$$\theta_c(t) = \frac{1}{N} \sum_{i=1}^{N} \theta_i(t) = \frac{1}{N} \sum_{i=1}^{N} \theta_i^0, \quad \forall\, t > 0. \tag{2.6}$$

Other than the simplified first-order model, in this paper we also consider the second-order dynamics with $X_i = 1$,

$$\text{Model B}: \quad \begin{cases} m_i \ddot{\theta}_i + \gamma_i \dot{\theta}_i = \Omega_i - \sum_{j=1}^{N} B_{ij} E_i E_j \sin(\theta_i - \theta_j) \\ \alpha_i \dot{E}_i = q_i - \beta_i E_i + \sum_{j=1}^{N} B_{ij} E_j \cos(\theta_i - \theta_j), \end{cases} \tag{2.7}$$

subject to initial data

$$\theta_i(0) = \theta_i^0, \quad \omega_i(0) := \dot{\theta}_i(0) = \omega_i^0, \quad E_i(0) = E_i^0, \tag{2.8}$$

and parameters

$$\sum_{i=1}^{N} \Omega_i = 0,\ \alpha_i > 0,\ q_i > 0,\ \beta_i \geq 1,\ B_{ij} = \begin{cases} 0, & i = j; \\ \geq 0, & i \neq j. \end{cases} \tag{2.9}$$

Model A and Model B actually fall into the category of generalized Kuramoto model with adaptive coupling. In [18, 20], several types of adaptive Kuramoto models were rigorously studied. However, to our best knowledge, the adaptive Kuramoto models in power grids has never been touched rigorously; in particular, this is the first rigorous study for a second-order Kuramoto model with adaptive couplings.

The ability of system to remain in synchronism (synchronization or phase-locking) is regarded as rotor angle stability, and the ability to maintain steady voltages (voltage stabilization) is regarded as voltage stability. The system (2.3) without dynamic voltages was considered in literature such as [8, 11, 12, 13, 25] where some conditions for the so-called "rotor angle stability" was found. However, the rigorous study of the system with dynamic voltages has not been touched. The following simulation illustrates that the voltages can be "unstable". We consider Model A with three oscillators where the parameters are set as

$$q = (1,2,3)^{\mathrm{T}}, \quad \beta = (3,6,9)^{\mathrm{T}}, \quad \Omega = (1.4, 1.8, -3.2)^{\mathrm{T}}, \quad B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 4 \\ 2 & 4 & 0 \end{pmatrix},$$

and the initial data is $\theta^0 = (0.6125, 0.9282, 0.7216)^{\mathrm{T}}$, $E^0 = (0.0791, 0.3592, 1.3103)^{\mathrm{T}}$. The following Figure 1 illustrates that the system fails to realize the synchronization and voltage stabilization.

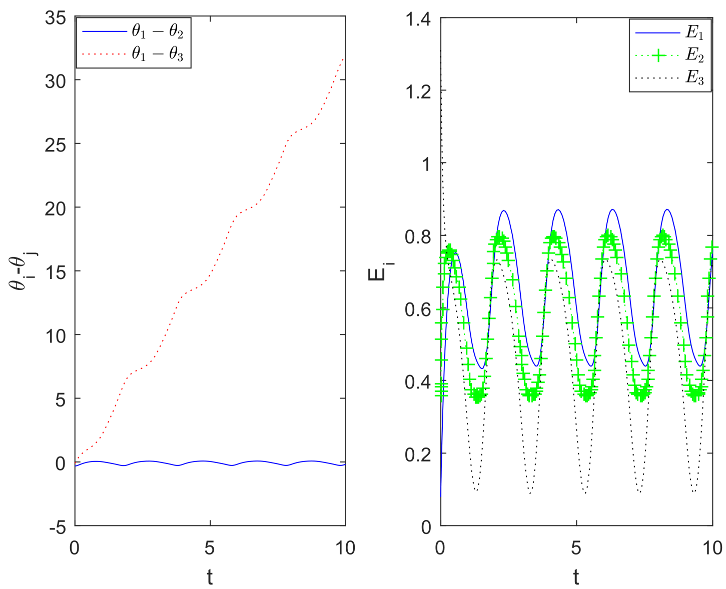


FIGURE 1. The phases are out of sync and voltages are unstable.

2.2. **Gradient formulation.** It is well-known that the first- or second-order Kuramoto model on a symmetric network is a gradient flow [9, 19]. Therefore, the power grid with constant coupling strength (i.e., constant voltage level) can be studied by the gradient approach, for example, [8, 25]. In this section, we point out that the coupled oscillators in power grid with dynamic voltages can be reformed as a gradient flow as well. We note that the topology of network is now varying in time, however, the underlying graph $\mathcal{G} = (\mathcal{V}, \mathcal{W})$ depends only on the matrix $B = (B_{ij})$ if the voltage levels are all positive. In this sense, we say the network topology is registered by the matrix $B$.

**Lemma 2.1.** *The system* (2.4) *or* (2.7) *is a gradient-like system with a real analytical potential* $V : \mathbb{R}^{2N} \to \mathbb{R}$, *i.e., there exists such a function* $V$ *such that* (2.4) *or* (2.7) *is reformulated as*

$$\dot{x} = -\nabla V(x) \quad or \quad M\ddot{x} + D\dot{x} = -\nabla V(x),$$

*if and only if the matrix* $B = (B_{ij})$ *is symmetric. Here,* $x := (\theta_1, \ldots, \theta_N, E_1, \ldots, E_N)^{\mathrm{T}}$, $M = \mathrm{diag}(m_1, m_2, \ldots, m_N, 0, 0, \ldots, 0)$ *and* $D = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_N, \alpha_1, \alpha_2, \ldots, \alpha_N)$.

*Proof.* (1) Suppose that the system (2.4) or (2.7) is a gradient-like system with a real analytic potential $V$, i.e.,

$$\frac{\partial V}{\partial \theta_i} = -\Omega_i + \sum_{j=1}^{N} B_{ij} E_i E_j \sin(\theta_i - \theta_j),$$

$$\frac{\partial V}{\partial E_i} = -q_i + \beta_i E_i - \sum_{j=1}^{N} B_{ij} E_j \cos(\theta_i - \theta_j), \quad i = 1, 2, \ldots, N.$$

Then, the analytic potential $V$ must satisfy

$$\frac{\partial^2 V}{\partial E_i \partial E_j} = \frac{\partial^2 V}{\partial E_j \partial E_i}, \ i \neq j,$$

i.e.,

$$B_{ij} \cos(\theta_i - \theta_j) = B_{ji} \cos(\theta_j - \theta_i), \ i \neq j.$$

This yields $B_{ij} = B_{ji}, i \neq j$, i.e., $B$ is symmetric.

(2) Suppose that the matrix $B$ is symmetric. We define $V : \mathbb{R}^{2N} \to \mathbb{R}$ as

$$V(x) := -\sum_{i=1}^{N} \Omega_i \theta_i - \sum_{i=1}^{N} q_i E_i + \frac{1}{2} \sum_{i=1}^{N} \beta_i E_i^2 - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} B_{ij} E_i E_j \cos(\theta_j - \theta_i).$$

Note that the function $V$ is real analytic in $(\theta, E)$, and it is easy to see that

$$\frac{\partial V}{\partial \theta_i} = -\Omega_i + \sum_{j=1}^{N} B_{ij} E_i E_j \sin(\theta_i - \theta_j),$$

$$\frac{\partial V}{\partial E_i} = -q_i + \beta_i E_i - \sum_{j=1}^{N} B_{ij} E_j \cos(\theta_i - \theta_j).$$

Therefore, system (2.4) or (2.7) is a gradient-like system:

$$\dot{x} = -\nabla V(x) \quad \text{or} \quad M\ddot{x} + D\dot{x} = -\nabla V(x).$$

□

For the first-order gradient system, we have the following result.

**Lemma 2.2.** [9, 19] *Let $x(\cdot)$ be a solution to the gradient-like system*

$$\dot{x} = -\nabla V(x),$$

*where $V : \mathbb{R}^{2N} \to \mathbb{R}$ is a real analytic function. If $x(\cdot) \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^{2N})$, then there exists an equilibrium $x_e \in \{x : \nabla V(x) = 0\}$ such that*

$$\lim_{t \to \infty} (\|\dot{x}(t)\| + \|x(t) - x_e\|) = 0.$$

For second-order gradient-like system with positive definite diagonal matrix $M$, the convergence result can be found in [25]. However, to deal with the model (2.7), we have to consider a degenerate second-order gradient-like system

$$M\ddot{x} + D\dot{x} = -\nabla V(x) \tag{2.10}$$

with a degenerate diagonal matrix $M$. The following theorem gives a convergence result for *degenerate* second-order gradient system. The proof relies on the quasi-gradient approach [26] and will be presented in Appendix.

**Theorem 2.1.** *Let $x = (\theta, E) : \mathbb{R}^+ \to \mathbb{R}^{2N}$ be a solution to the gradient-like system* (2.10) *with analytic potential $V : \mathbb{R}^{2N} \to \mathbb{R}$. If*

$$\theta(\cdot) \in \ell^{1,\infty}(\mathbb{R}^+, \mathbb{R}^N) \quad \text{and} \quad E(\cdot) \in \ell^{\infty}(\mathbb{R}^+, \mathbb{R}^N), \tag{2.11}$$

*then there exists an equilibrium $x_e = (\theta_e, E_e) \in \{x \,|\, \nabla V(x) = 0\}$ such that*

$$\lim_{t\to\infty} (\|\dot{x}(t)\| + \|x(t) - x_e\|) = 0.$$

*Proof.* See Appendix. □

**Remark 2.1.** Due to the periodicity of the sine function, the phase variables of (2.4) or (2.7) may be considered as living on a torus. However, in order to apply the gradient approach in the analysis of (2.4) or (2.7), we need to consider these variables as living on a Euclidean space, i.e., $(\theta(t), E(t)) \in \mathbb{R}^N \times \mathbb{R}^N$. The reason is that the potential function becomes discontinuous if we consider the phase variables as living on a torus.

2.3. **Elementary estimates.** In this part we give an estimate for the positivity of voltages $E_i$'s.

**Lemma 2.3.** *Let $(\theta(t), E(t))$ be a solution to the coupled system* (2.4) *or* (2.7) *satisfying*

$$\min_{1\le i\le N} E_i^0 > 0, \qquad \sup_{t\in[0,T^*)} \max_{1\le i,j\le N} |\theta_i(t) - \theta_j(t)| < \frac{\pi}{2}. \tag{2.12}$$

*Then, we have*

$$\min_{1\le i\le N} E_i(t) > 0, \quad \forall\, t \in [0, T^*).$$

*Proof.* We define the set $\mathcal{T}$:

$$\mathcal{T} := \left\{ t \in [0, T^*) \;\middle|\; \min_{1\le i\le N} E_i(t) = 0 \right\}.$$

We now claim that $\mathcal{T} = \emptyset$. Suppose to the contrary that $\mathcal{T} \neq \emptyset$, and we define $\mathcal{T}^* := \inf \mathcal{T}$. Note that since $\min_{1\le i\le N} E_i^0 > 0$ and $\min_{1\le i\le N} E_i(t)$ is a continuous function of $t$, then we have

$$\min_{1\le i\le N} E_i(t) > 0,\ t \in [0, \mathcal{T}^*), \quad \min_{1\le i\le N} E_i(\mathcal{T}^*) = 0.$$

Thus, there exists $i_0 \in \{1, 2, \ldots, N\}$ such that $E_{i_0}(\mathcal{T}^*) = \min_{1\le i\le N} E_i(\mathcal{T}^*) = 0$ and

$$E_j(\mathcal{T}^*) \ge E_{i_0}(\mathcal{T}^*) = 0,\ j = 1, 2, \ldots, N. \tag{2.13}$$

Then we have

$$\alpha_{i_0} \frac{d}{dt} E_{i_0}(\mathcal{T}^*) = q_{i_0} - \beta_{i_0} E_{i_0}(\mathcal{T}^*) + \sum_{j=1}^N B_{i_0 j} E_j(\mathcal{T}^*) \cos(\theta_{i_0}(\mathcal{T}^*) - \theta_j(\mathcal{T}^*)) \ge q_{i_0} > 0.$$

where we used (2.13) and (2.12). This implies that there exists $\delta > 0$ such that

$$E_{i_0}(t) < E_{i_0}(\mathcal{T}^*) = 0,\ \ t \in (\mathcal{T}^* - \delta, \mathcal{T}^*),$$

which is contradictory to $E_{i_0}(t) \geq \min_{1\leq i\leq N} E_i(t) > 0,\ t \in [0, \mathcal{T}^*)$. Therefore, we have $\mathcal{T} = \emptyset$. Again, we use (2.12) to obtain

$$\min_{1\leq i\leq N} E_i(t) > 0, \quad \forall\, t \in [0, T^*).$$

□

We denote the underlying graph of system (2.4) or (2.7) by $\mathcal{G} = (\mathcal{V}, \mathcal{W})$. We note that the underlying network is undirected, i.e., the matrix $B = (B_{ij})$ is symmetric. We say a graph $\mathcal{G}$ is connected if for any pair of nodes $i, j \in \mathcal{V}$, there exists a shortest path from $i$ to $j$, say

$$i = p_1 \to p_2 \to p_3 \to \cdots \to p_{d_{ij}} = j, \quad (p_k, p_{k+1}) \in \mathcal{W}, \quad k = 1, 2, \ldots, d_{ij} - 1.$$

Here, $d_{ij}$ denotes the *distance* between $i$ and $j$, i.e., the minimum length of paths connecting $i$ and $j$. We also denote the *diameter* of graph $\mathcal{G}$ by $\mathrm{d}(\mathcal{G})$, i.e., $\mathrm{d}(\mathcal{G}) := \max\{d_{ij} | 1 \leq i, j \leq N\}$. In order for the complete synchronization, we assume that the underlying graph $\mathcal{G}$ is connected. The following result, which connects the total deviations and the partial deviations along the edges in a connected graph, will be useful in the energy estimate.

**Lemma 2.4.** [9] *Suppose that the graph $\mathcal{G} = (\mathcal{V}, \mathcal{W})$ is connected and let $\theta_i$ be the phase of the i-th oscillators. Then, for any ensemble of phases $(\theta_1, \theta_2, \ldots, \theta_N)$, we have*

$$L_* \sum_{l=1}^N \sum_{k=1}^N |\theta_l - \theta_k|^2 \leq \sum_{(l,k)\in\mathcal{W}} |\theta_l - \theta_k|^2 \leq \sum_{l=1}^N \sum_{k=1}^N |\theta_l - \theta_k|^2, \quad \textit{with } L_* := \frac{1}{1 + \mathrm{d}(\mathcal{G})|\mathcal{W}^c|}.$$

*Here $\mathcal{W}^c$ is the complement of edge set $\mathcal{W}$ in $\mathcal{V} \times \mathcal{V}$ and $|\mathcal{W}^c|$ denotes its cardinality.*

2.4. **Main results.** In this subsection, we present the main results of this paper. Before the statement we lay out some notation used in the following context:

$$B_l := \min_{(i,j)\in\mathcal{W}} B_{ij}, \quad B_u := \max_{(i,j)\in\mathcal{W}} B_{ij}, \quad \alpha_l := \min_{1\leq i\leq N} \alpha_i,$$
$$m_l := \min_{1\leq i\leq N} m_i, \quad m_u := \max_{1\leq i\leq N} m_i, \quad \gamma_l := \min_{1\leq i\leq N} \gamma_i, \quad \gamma_u := \max_{1\leq i\leq N} \gamma_i,$$
$$\hat{\gamma}_i := \gamma_i - \frac{1}{N}\sum_{j=1}^N \gamma_j, \quad \hat{m}_i := m_i - \frac{1}{N}\sum_{j=1}^N m_j, \quad \theta_c(t) = \frac{1}{N}\sum_{i=1}^N \theta_i(t),$$
$$D(\theta(t)) := \max_{1\leq i,j\leq N} |\theta_i(t) - \theta_j(t)|, \quad D(\Omega) := \max_{1\leq i,j\leq N} |\Omega_i - \Omega_j|, \quad E_* := \min_{1\leq i\leq N} \Big\{E_i^0, \frac{q_i}{\beta_i}\Big\},$$
$$K := -\max_{1\leq i\leq N} \Big\{ -\beta_i + \sum_{j=1}^N B_{ij}\Big\}, \quad E^* := \max\Big\{\frac{\sum_{i=1}^N q_i}{K},\ \sum_{i=1}^N E_i^0\Big\},$$
$$K_\alpha := -\max_{1\leq i\leq N}\Big\{\frac{-\beta_i + \sum_{j=1}^N B_{ij}}{\alpha_i}\Big\}, \quad E_\alpha^* := \max\Big\{\frac{\sum_{i=1}^N q_i}{\alpha_l K_\alpha},\ \frac{\sum_{i=1}^N \alpha_i E_i^0}{\alpha_l}\Big\}.$$

2.4.1. *First-order model.* For identical oscillators, we consider the system (2.4) on a connected network, i.e., the underlying graph $\mathcal{G}$ registered by the matrix $B = (B_{ij})$ is connected. We have the following result.

**Theorem 2.2.** *(Identical oscillators) Let $(\theta(t), E(t))$ be a solution to the system* (2.4)-(2.5). *If*

*(1)* $\Omega_i = 0, i = 1, 2, \dots, N$,
*(2)* $\min\limits_{1\le i\le N} E_i^0 > 0,\ D(\theta^0) < \frac{\pi}{2}$,
*(3)* $\max\limits_{1\le i\le N} \left\{-\beta_i + \sum_{j=1}^N B_{ij}\right\} < 0.$

*Then we have asymptotic phase synchronization*

$$\|\theta^0 - \theta_c^0\mathbb{1}\|e^{-NB_uE^{*2}t} \le \|\theta(t) - \theta_c^0\mathbb{1}\| \le \|\theta^0 - \theta_c^0\mathbb{1}\|e^{-\frac{NL_*B_lE_*^2\sin D(\theta^0)}{D(\theta^0)}t},\ \ t > 0.$$

*Furthermore, there exists* $E^\infty \in \mathbb{R}_+^N$ *such that*

$$\lim_{t\to\infty}\left(\|\dot{E}(t)\| + \|E(t) - E^\infty\|\right) = 0.$$

*Here,* $\theta_c^0 = \frac{1}{N}\sum_{i=1}^N \theta_i^0$, $\mathbb{1} := (1, 1, \dots, 1)^{\mathrm{T}}$.

For nonidentical oscillators, we introduce a parameter

$$B_* := \min_{1\le i\ne j\le N}\left\{2B_{ij} + \sum_{k\ne i,j}\min\{B_{ik}, B_{jk}\}\right\}.$$

Note that

$$\begin{aligned} \mathrm{d}(\mathcal{G}) \le 2 \quad &\Longleftrightarrow \quad \forall i, j,\ \ B_{ij} \ne 0 \quad \text{or} \quad \exists k \ne i, j, s.t. \min\{B_{ik}, B_{jk}\} > 0 \\ &\Longleftrightarrow \quad \forall i, j,\ \ 2B_{ij} + \sum_{k\ne i,j}\min\{B_{ik}, B_{jk}\} > 0 \\ &\Longleftrightarrow \quad \min_{1\le i\ne j\le N}\left\{2B_{ij} + \sum_{k\ne i,j}\min\{B_{ik}, B_{jk}\}\right\} =: B_* > 0. \end{aligned}$$

In the following theorem, we need to suppose $B_* > 0$. Therefore, we consider the system (2.4) with restricted connectivity that $\mathrm{d}(\mathcal{G}) \le 2$ in the following theorem.

**Theorem 2.3.** *(Nonidentical oscillators) Let* $(\theta(t), E(t))$ *be a solution to the system* (2.4)-(2.5). *If*

*(1)* $D(\Omega) > 0,\ \mathrm{d}(\mathcal{G}) \le 2,$ *and* $D(\Omega) < B_*E_*^2$,
*(2)* $\min\limits_{1\le i\le N} E_i^0 > 0,\ D(\theta^0) < \frac{\pi}{2}$,
*(3)* $\max\limits_{1\le i\le N} \left\{-\beta_i + \sum_{j=1}^N B_{ij}\right\} < 0.$

*Then, there exist* $(\theta^\infty, E^\infty) \in \left\{x \in \mathbb{R}^{2N} \,\middle|\, \nabla V(x) = 0\right\}$ *such that*

$$\lim_{t\to\infty}\left(\|\dot{\theta}(t)\| + \|\theta(t) - \theta^\infty\|\right) = 0, \quad \lim_{t\to\infty}\left(\|\dot{E}(t)\| + \|E(t) - E^\infty\|\right) = 0.$$

2.4.2. *Second-order model.* We first introduce our main hypotheses on the parameters and initial configurations below.

**(H1)** the underlying graph $\mathcal{G}$ is connected.

**(H2)** Let $D_0 \in (0, \frac{\pi}{2})$ be given. The initial voltages and parameters satisfy

$$\min_{1\le i\le N} E_i^0 > 0, \quad \max_{1\le i\le N}\left\{-\beta_i + \sum_{j=1}^N B_{ij}\right\} < 0, \quad \frac{2NB_lL_*E_*^2\sin D_0}{D_0} > \lambda,$$

and

$$\frac{2\kappa NE_\alpha^*B_uD_0}{2NB_lL_*E_*^2\sin D_0 - \lambda D_0} < \frac{\gamma_l}{2m_u + \lambda},$$

where $\hat{\Gamma} := \mathrm{diag}\{\hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_N\}, \hat{M} := \mathrm{diag}\{\hat{m}_1, \hat{m}_2, \dots, \hat{m}_N\}$, and $\lambda := \frac{\sqrt{\mathrm{tr}(\hat{\Gamma}^2)}+2\sqrt{\mathrm{tr}(\hat{M}^2)}}{\sqrt{N}}$ with $\mathrm{tr}(\cdot)$ denoting the trace of a matrix.

**(H3)** For some $\varepsilon, \mu > 0$ with

$$\frac{\varepsilon}{\mu} \in \left( \frac{2\kappa N E_\alpha^* B_u D_0}{2NB_l L_* E_*^2 \sin D_0 - \lambda D_0}, \frac{\gamma_l}{2m_u + \lambda} \right), \quad \text{and} \quad \frac{2\sqrt{2}C_2 \max\{\varepsilon, \mu\}\|\Omega\|}{C_1 C_L} < \frac{1}{2}D_0,$$

the initial data $\theta^0 = (\theta_i^0, \dots, \theta_N^0)$ and $\omega^0 = (\omega_i^0, \dots, \omega_N^0)$ satisfy

$$\sqrt{\frac{\mathcal{E}[\theta^0, \omega^0]}{C_1}} < \frac{1}{2}D_0.$$

Here

$$C_1 := \min\left\{ \varepsilon\gamma_l - \frac{2\varepsilon^2 m_u}{\mu} + \frac{2\mu N B_l L_* E_*^2 (1-\cos D_0)}{D_0^2},\ \frac{\mu m_l}{2} \right\},$$

$$C_2 := \max\left\{ \varepsilon\gamma_u + \frac{2\varepsilon^2 m_u \gamma_u}{\mu\gamma_l} + \mu N B_u {E_\alpha^*}^2,\ \frac{3\mu m_u}{2} \right\}, \quad \kappa := \frac{\max\limits_{1\le i\le N}\{q_i - \beta_i E_*\} + NB_u E_\alpha^*}{\alpha_l},$$

$$C_L := \min\left\{ 2(\mu\gamma_l - \varepsilon m_u), \frac{2\varepsilon N B_l L_* E_*^2 \sin D_0}{D_0} - 2\kappa\mu N E_\alpha^* B_u \right\} - \varepsilon\lambda,$$

and

$$\mathcal{E}[\theta, \omega] := \varepsilon\sum_{i=1}^N \gamma_i(\theta_i - \theta_c)^2 + 2\varepsilon\sum_{i=1}^N m_i(\theta_i-\theta_c)\omega_i + \mu\sum_{i=1}^N m_i\omega_i^2 + \mu\sum_{(i,j)\in\mathcal{W}} B_{ij}E_iE_j\left(1-\cos(\theta_i-\theta_j)\right).$$

The main result in this part is as follows.

**Theorem 2.4.** *Let $(\theta(t), E(t))$ be a solution to the coupled system* (2.7)-(2.9) *satisfying hypotheses* **(H1)-(H3)**. *Then, there exists $(\theta^\infty, E^\infty) \in \left\{ x \in \mathbb{R}^{2N} \,\middle|\, \nabla V(x) = 0 \right\}$ such that*

$$\lim_{t\to\infty}\left(\|\dot{\theta}(t)\| + \|\theta(t) - \theta^\infty\|\right) = 0, \quad \lim_{t\to\infty}\left(\|\dot{E}(t)\| + \|E(t) - E^\infty\|\right) = 0.$$

## 3. Synchronization of first-order power grids

In this section, we study an emergent dynamics of model (2.4)-(2.5) which has been motivated by recent observations in the power system. Below, we give the dynamic properties of synchronization estimates for identical and nonidentical oscillators.

3.1. **Synchronization: Identical oscillators.** We consider the synchronization of the ensemble of identical oscillators. For identical oscillators, the system (2.4)-(2.5) becomes

$$\begin{cases} \dot{\theta}_i = -\displaystyle\sum_{j=1}^N B_{ij}E_iE_j\sin(\theta_i - \theta_j) \\ \dot{E}_i = q_i - \beta_i E_i + \displaystyle\sum_{j=1}^N B_{ij}E_j\cos(\theta_i-\theta_j), \quad i = 1, 2, \dots, N, \end{cases} \tag{3.1}$$

subject to

$$\theta_i(0) = \theta_i^0,\ \ E_i(0) = E_i^0,\ \ q_i > 0,\ \ \beta_i \geq 1,\ \ B_{ij} = \begin{cases} 0, & i = j; \\ \geq 0, & i \neq j. \end{cases} \tag{3.2}$$

We introduce simplified notations:

$$\theta_M(t) := \max_{1\leq i\leq N} \theta_i(t), \quad \theta_m(t) := \min_{1\leq i\leq N} \theta_i(t).$$

**Lemma 3.1.** *Let* $(\theta(t), E(t))$ *be a solution to the coupled system* (3.1)-(3.2) *with*

$$\min_{1\leq i\leq N} E_i^0 > 0, \quad D(\theta^0) < \frac{\pi}{2}.$$

*Then, for any* $t \geq 0$ *we have*

$$\min_{1\leq i\leq N} E_i(t) > 0, \quad D(\theta(t)) \leq D(\theta^0).$$

*Proof.* We define the set $\mathcal{T}$ and its supremum:

$$\mathcal{T} := \left\{ T \in [0, +\infty) \,\middle|\, D(\theta(t)) < \frac{\pi}{2},\ \forall\, t \in [0, T) \right\}, \quad T^* := \sup \mathcal{T}.$$

Owing to the initial condition $D(\theta^0) < \frac{\pi}{2}$, and the continuity of $D(\theta(t))$ with respect to $t$, the set $\mathcal{T}$ is not empty and $T^*$ is well-defined. We now claim that

$$T^* = \infty.$$

Suppose to the contrary that $T^* < \infty$. Then we have

$$D(\theta(t)) < \frac{\pi}{2},\ t \in [0, T^*), \quad \text{and} \quad D(\theta(T^*)) = \frac{\pi}{2}. \tag{3.3}$$

By Lemma 2.3 and (3.2), we obtain

$$\begin{aligned} \frac{d}{dt} D(\theta(t)) &= \dot{\theta}_M(t) - \dot{\theta}_m(t) \\ &= -\sum_{j=1}^N B_{Mj} E_i^M(t) E_j(t) \sin(\theta_M(t) - \theta_j(t)) + \sum_{j=1}^N B_{mj} E_i^m(t) E_j(t) \sin(\theta_m(t) - \theta_j(t)) \\ &\leq 0, \quad t \in [0, T^*), \end{aligned}$$

which implies thatp $D(\theta(t)) \leq D(\theta^0) < \frac{\pi}{2},\ t \in [0, T^*)$. Let $t \to T^{*-}$ and we obtain

$$D(\theta(T^*)) < \frac{\pi}{2},$$

which is contradictory to (3.3). This proves the claim that $T^* = \infty$. Repeating Lemma 2.3 and the above steps again, we get

$$D(\theta(t)) \leq D(\theta^0), \quad \min_{1\leq i\leq N} E_i(t) > 0,\ \forall\, t \geq 0.$$

□

**Lemma 3.2.** *Let* $(\theta(t), E(t))$ *be a solution to the coupled system* (3.1)-(3.2) *with*

$$\min_{1\leq i\leq N} E_i^0 > 0, \quad D(\theta^0) < \frac{\pi}{2}.$$

*Then, we have*

$$\min_{1\leq i\leq N} E_i(t) \geq E_*, \quad \forall\, t \geq 0.$$

*Proof.* It follows from Lemma 3.1 that

$$\frac{d}{dt}E_i(t) = q_i - \beta_i E_i(t) + \sum_{j=1}^{N} B_{ij}E_j(t)\cos(\theta_i(t) - \theta_j(t)) \geq q_i - \beta_i E_i(t),\ t \geq 0.$$

By the comparison principle, we obtain

$$\min_{1\leq i\leq N} E_i(t) \geq E_*,\ \forall t \geq 0.$$

□

**Lemma 3.3.** *Let* $(\theta(t), E(t))$ *be a solution to the coupled system* (3.1)-(3.2). *If*

*(1)* $\min\limits_{1\leq i\leq N} E_i^0 > 0,\ D(\theta^0) < \frac{\pi}{2}$,

*(2)* $\max\limits_{1\leq i\leq N} \left\{-\beta_i + \sum_{j=1}^{N} B_{ij}\right\} < 0$.

*Then, we have*

$$\sum_{i=1}^{N} E_i(t) \leq E^*,\ t \geq 0.$$

*Proof.* It follows from Lemma 3.1 that

$$\sup_{t\geq 0} D(\theta(t)) \leq D(\theta^0), \quad \min_{1\leq i\leq N} E_i(t) > 0,\ \forall t \geq 0.$$

We consider the temporal evolution of $E_i(t)$:

$$\frac{d}{dt}E_i(t) = q_i - \beta_i E_i(t) + \sum_{j=1}^{N} B_{ij}E_j(t)\cos(\theta_i(t) - \theta_j(t)) \leq q_i - \beta_i E_i(t) + \sum_{j=1}^{N} B_{ij}E_j(t),\ t \geq 0.$$

Hence,

$$\begin{aligned}
\frac{d}{dt}\sum_{i=1}^{N} E_i(t) &\leq \sum_{i=1}^{N} q_i - \sum_{i=1}^{N} \beta_i E_i(t) + \sum_{i=1}^{N}\sum_{j=1}^{N} B_{ij}E_j(t) \\
&= \sum_{i=1}^{N} q_i + \sum_{i=1}^{N} \left(-\beta_i + \sum_{j=1}^{N} B_{ij}\right) E_i(t) \\
&\leq \sum_{i=1}^{N} q_i - K\sum_{i=1}^{N} E_i(t),\ t \geq 0,
\end{aligned}$$

here, $K$ is defined in Theorem 2.2 and we use the symmetry of $B = (B_{ij})$ and

$$\sum_{i=1}^{N}\sum_{j=1}^{N} B_{ij}E_j(t) = \sum_{j=1}^{N}\sum_{i=1}^{N} B_{ji}E_i(t) = \sum_{i=1}^{N}\sum_{j=1}^{N} B_{ij}E_i(t).$$

By the comparison principle,

$$\sum_{i=1}^{N} E_i(t) \leq E^*,\ t \geq 0.$$

□

**Remark 3.1.** *By Lemmas 3.2 and 3.3, under the conditions of Lemma 3.3, $E_i$'s are bounded by $E_*$ and $E^*$:*

$$E_* \leq E_i(t) \leq E^*, \ t \geq 0.$$

For simplicity in the estimate, we assume that $\sum_{i=1}^N \theta_i^0 = 0$ in the following lemma.

**Lemma 3.4.** *Let $(\theta(t), E(t))$ be a solution to the system* (3.1)-(3.2). *If*

*(1)* $\min\limits_{1\leq i\leq N} E_i^0 > 0, \ D(\theta^0) < \frac{\pi}{2}, \ \sum_{i=1}^N \theta_i^0 = 0,$

*(2)* $\max\limits_{1\leq i\leq N} \left\{-\beta_i + \sum_{j=1}^N B_{ij}\right\} < 0.$

*Then, we have*

$$-NB_uE^{*2}\|\theta(t)\| \leq \frac{d}{dt}\|\theta(t)\| \leq -\frac{NL_*B_lE_*^2 \sin D(\theta^0)}{D(\theta^0)}\|\theta(t)\|, \ t \geq 0.$$

*Proof.* A straightforward computation yields

$$\begin{aligned}
\frac{d}{dt}\|\theta(t)\|^2 &= 2\|\theta(t)\|\frac{d}{dt}\|\theta(t)\| = 2\sum_{i=1}^N \theta_i(t)\dot{\theta}_i(t) \\
&= 2\sum_{i=1}^N \theta_i(t)\Bigg(-\sum_{j=1}^N B_{ij}E_i(t)E_j(t)\sin(\theta_i(t)-\theta_j(t))\Bigg) \\
&= -2\sum_{(i,j)\in\mathcal{W}} B_{ij}E_i(t)E_j(t)\theta_i(t)\sin(\theta_i(t)-\theta_j(t)) \\
&= -\sum_{(i,j)\in\mathcal{W}} B_{ij}E_i(t)E_j(t)(\theta_i(t)-\theta_j(t))\sin(\theta_i(t)-\theta_j(t)).
\end{aligned}$$

• Step A (Lower bound): We use the relation

$$\sum_{i=1}^N \theta_i^0 = 0, \quad (\theta_i(t)-\theta_j(t))\sin(\theta_i(t)-\theta_j(t)) \leq |(\theta_i(t)-\theta_j(t))\sin(\theta_i(t)-\theta_j(t))| \leq |\theta_i(t)-\theta_j(t)|^2$$

and Lemma 2.4, Remark 3.1 to obtain

$$\begin{aligned}
2\|\theta(t)\|\frac{d}{dt}\|\theta(t)\| &\geq -\sum_{(i,j)\in\mathcal{W}} B_{ij}E_i(t)E_j(t)|\theta_i(t)-\theta_j(t)|^2 \geq -B_uE^{*2}\sum_{(i,j)\in\mathcal{W}}|\theta_i(t)-\theta_j(t)|^2 \\
&\geq -B_uE^{*2}\sum_{i=1}^N\sum_{j=1}^N|\theta_i(t)-\theta_j(t)|^2 = -2NB_uE^{*2}\|\theta(t)\|^2, \ t \geq 0.
\end{aligned}$$

Thus,

$$\frac{d}{dt}\|\theta(t)\| \geq -NB_uE^{*2}\|\theta(t)\|, \ t \geq 0.$$

• Step B (Upper bound): We use the relation

$$\sum_{i=1}^N \theta_i^0 = 0, \quad (\theta_i(t)-\theta_j(t))\sin(\theta_i(t)-\theta_j(t)) \geq \frac{\sin D(\theta^0)}{D(\theta^0)}(\theta_i(t)-\theta_j(t))^2$$

and Lemma 2.4, Remark 3.1 to obtain

$$2\|\theta(t)\|\frac{d}{dt}\|\theta(t)\|\leq -\frac{\sin D(\theta^0)}{D(\theta^0)}\sum_{(i,j)\in\mathcal{W}} B_{ij}E_i(t)E_j(t)|\theta_i(t)-\theta_j(t)|^2\leq -\frac{B_l E_*^2\sin D(\theta^0)}{D(\theta^0)}\sum_{(i,j)\in\mathcal{W}}|\theta_i(t)-\theta_j(t)|^2$$

$$\leq -\frac{L_* B_l E_*^2\sin D(\theta^0)}{D(\theta^0)}\sum_{i=1}^N\sum_{j=1}^N|\theta_i(t)-\theta_j(t)|^2= -\frac{2NL_* B_l E_*^2\sin D(\theta^0)}{D(\theta^0)}\|\theta(t)\|^2,\ t\geq 0.$$

Thus,

$$\frac{d}{dt}\|\theta(t)\|\leq -\frac{NL_* B_l E_*^2\sin D(\theta^0)}{D(\theta^0)}\|\theta(t)\|,\ t\geq 0.$$

The proof is completed. □

Now we give a proof for Theorem 2.2.

*Proof of Theorem 2.2.* If $\theta_c^0=\frac{1}{N}\sum_{i=1}^N\theta_i^0=0$, we use Lemma 3.4 and Gronwall's inequality to find that

$$\|\theta^0\|e^{-NB_uE^{*2}t}\leq\|\theta(t)\|\leq\|\theta^0\|e^{-\frac{NL_*B_l\sin D(\theta^0)E_*^2}{D(\theta^0)}t},\ t\geq 0.$$

If $\theta_c^0=\frac{1}{N}\sum_{i=1}^N\theta_i^0\neq 0$, we make a variable change

$$\hat{\theta}_i(t)=\theta_i(t)-\theta_c(t)=\theta_i(t)-\theta_c^0$$

and use new variables $\hat{\theta}_i(t)$ to replace $\theta_i(t)$. Then we find that Lemma 3.4 is available for $\hat{\theta}=(\hat{\theta}_1,\dots,\hat{\theta}_N)$, i.e.,

$$-NB_uE^{*2}\|\hat{\theta}(t)\|\leq\frac{d}{dt}\|\hat{\theta}(t)\|\leq -\frac{NL_*B_lE_*^2\sin D(\hat{\theta}^0)}{D(\hat{\theta}^0)}\|\hat{\theta}(t)\|,\ t\geq 0.$$

This is the first assertion in Theorem 2.2. For the second assertion we use the gradient flow approach. Note that the boundedness of $\theta(t)$ and $E(t)$ simply follows from the first assertion and Remark 3.1, then we obtain the desired result by Lemma 2.1 and Lemma 2.2. □

3.2. **Nonidentical oscillators.** We now consider the ensemble of nonidentical oscillators, i.e., (2.4)-(2.5).

**Lemma 3.5.** *Let* $(\theta(t),E(t))$ *be a solution to the coupled system* (2.4)-(2.5). *If*

*(1)* $D(\Omega)>0,\ B_*>0,\ D(\Omega)<B_*E_*^2$,
*(2)* $\min_{1\leq i\leq N}E_i^0>0,\ D(\theta^0)<\frac{\pi}{2}$.

*Then, for all* $t>0$ *we have*

$$\min_{1\leq i\leq N}E_i(t)\geq E_*,\quad \textit{and}\quad D(\theta(t))<\frac{\pi}{2}.$$

*Proof.* We define the set $\mathcal{T}$ and its supremum:

$$\mathcal{T}:=\left\{T\in[0,+\infty)\,\middle|\,D(\theta(t))<\frac{\pi}{2},\ \forall\, t\in[0,T)\right\},\quad T^*:=\sup\mathcal{T}.$$

Note that since $D(\theta^0)<\frac{\pi}{2}$, and $D(\theta(t))$ is a continuous function of $t$, there exists $\eta>0$ such that

$$D(\theta(t))<\frac{\pi}{2},\ t\in[0,\eta).$$

Therefore, the set $\mathcal{T}$ is not empty, and $T^*$ is well-defined. We now claim that

$$T^* = \infty.$$

Suppose to the contrary that $T^* < \infty$. Then from the continuity of $D(\theta(t))$,

$$D(\theta(t)) < \frac{\pi}{2},\ t \in [0, T^*), \quad D(\theta(T^*)) = \frac{\pi}{2}. \tag{3.4}$$

We use (2.5), (3.4) and Lemma 2.3 to obtain

$$\frac{d}{dt}E_i(t) = q_i - \beta_i E_i(t) + \sum_{j=1}^N B_{ij} E_j(t) \cos(\theta_i(t) - \theta_j(t)) \geq q_i - \beta_i E_i(t), \quad t \in [0, T^*).$$

By the comparison principle, we obtain for any $i \in \{1, 2, \dots, N\}$

$$E_i(t) \geq E_*, \quad t \in [0, T^*).$$

We now consider the temporal evolution of $D(\theta(t))$:

$$\begin{aligned}
\frac{d}{dt}D(\theta(t)) &= \dot{\theta}_M(t) - \dot{\theta}_m(t) \\
&= \Omega_M - \sum_{j=1}^N B_{Mj} E_i^M(t) E_j(t) \sin(\theta_M(t) - \theta_j(t)) - \Omega_m + \sum_{j=1}^N B_{mj} E_i^m(t) E_j(t) \sin(\theta_m(t) - \theta_j(t)) \\
&\leq D(\Omega) - E_*^2 \left[ \sum_{j=1}^N B_{Mj} \sin(\theta_M(t) - \theta_j(t)) - \sum_{j=1}^N B_{mj} \sin(\theta_m(t) - \theta_j(t)) \right] \\
&= D(\Omega) - E_*^2 \left[ 2B_{Mm} \sin D(\theta(t)) + \sum_{j \neq M,m} \min\{B_{Mj}, B_{mj}\} \left( \sin(\theta_M(t) - \theta_j(t)) + \sin(\theta_j(t) - \theta_m(t)) \right) \right] \\
&= D(\Omega) - E_*^2 \left[ 2B_{Mm} \sin D(\theta(t)) + 2 \sin \frac{D(\theta(t))}{2} \sum_{j \neq M,m} \min\{B_{Mj}, B_{mj}\} \cos(\frac{\theta_M(t) + \theta_m(t)}{2} - \theta_j(t)) \right] \\
&\leq D(\Omega) - E_*^2 \left[ 2B_{Mm} + \sum_{j \neq M,m} \min\{B_{Mj}, B_{mj}\} \right] \sin D(\theta(t)) \\
&\leq D(\Omega) - E_*^2 \min_{1 \leq i \neq k \leq N} \left\{ 2B_{ik} + \sum_{j \neq i,k} \min\{B_{ij}, B_{kj}\} \right\} \sin D(\theta(t)) \\
&\leq D(\Omega) - \frac{2B_* E_*^2}{\pi} D(\theta(t)), \quad t \in [0, T^*).
\end{aligned}$$

Here $E_i^M \in \{E_i | \theta_i = \theta_M\}$, $E_i^m \in \{E_i | \theta_i = \theta_m\}$ and we used

$$-\frac{D(\theta(t))}{2} \leq \frac{\theta_m(t) - \theta_j(t)}{2} \leq 0 \leq \frac{\theta_M(t) - \theta_j(t)}{2} \leq \frac{D(\theta(t))}{2},\ t \in [0, T^*),$$

and

$$\sin y \geq \frac{2}{\pi} y, \quad y \in [0, \frac{\pi}{2}].$$

Hence, we obtain

$$D(\theta(t)) \le \max\left\{\frac{\pi D(\Omega)}{2B_* E_*^2}, D(\theta^0)\right\} < \frac{\pi}{2}, \qquad t \in [0, T^*).$$

Let $t \to T^{*-}$ and we have $D(\theta(T^*)) < \frac{\pi}{2}$, which is contradictory to (3.4). Therefore, we have

$$T^* = \infty.$$

We now repeat the above steps again for $t \in [0, +\infty)$ to find that

$$\min_{1\le i\le N} E_i(t) \ge E_*, \qquad D(\theta(t)) < \tfrac{\pi}{2}.$$

□

**Lemma 3.6.** *Let* $(\theta(t), E(t))$ *be a solution to the coupled system* (2.4)-(2.5).. *If*

*(1)* $D(\Omega) > 0,\ B_* > 0,\ D(\Omega) < B_* E_*^2,$

*(2)* $\min\limits_{1\le i\le N} E_i^0 > 0,\ D(\theta^0) < \frac{\pi}{2},$

*(3)* $\max\limits_{1\le i\le N} \left\{-\beta_i + \sum_{j=1}^N B_{ij}\right\} < 0.$

*then we have*

$$\sum_{i=1}^N E_i(t) \le E^*,\ t \ge 0.$$

*Proof.* It follows from Lemma 3.5 and (2.5) that for any $i \in \{1, 2, \dots, N\}$

$$\begin{aligned}\frac{d}{dt}E_i(t) &= q_i - \beta_i E_i(t) + \sum_{j=1}^N B_{ij} E_j(t)\cos(\theta_i(t) - \theta_j(t)) \\ &\le q_i - \beta_i E_i(t) + \sum_{j=1}^N B_{ij} E_j(t),\ t \ge 0.\end{aligned}$$

We have

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^N E_i(t) &\le \sum_{i=1}^N q_i - \sum_{i=1}^N \beta_i E_i(t) + \sum_{i=1}^N\sum_{j=1}^N B_{ij}E_j(t) \\ &= \sum_{i=1}^N q_i + \sum_{i=1}^N \left(-\beta_i + \sum_{j=1}^N B_{ij}\right) E_i(t) \\ &\le \sum_{i=1}^N q_i - K\sum_{i=1}^N E_i(t),\ t \ge 0.\end{aligned}$$

By the comparison principle, we obtain

$$\sum_{i=1}^N E_i(t) \le E^*,\ t \ge 0.$$

□

**Proposition 3.1.** *Let* $(\theta(t), E(t))$ *be a solution to the coupled system* (2.4)-(2.5). *If*

*(1)* $D(\Omega) > 0,\ B_* > 0,\ D(\Omega) < B_* E_*^2$,
*(2)* $\min\limits_{1 \le i \le N} E_i^0 > 0,\ D(\theta^0) < \frac{\pi}{2}$,
*(3)* $\max\limits_{1 \le i \le N} \left\{ -\beta_i + \sum_{j=1}^N B_{ij} \right\} < 0$.

*then we have*

$$E(\cdot) \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^N), \quad \theta(\cdot) \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^N).$$

*Proof.* It follows from Lemma 3.5 and Lemma 3.6 that

$$\sup_{t \ge 0} D(\theta(t)) \le \frac{\pi}{2},\ E_* \le E_i(t) \le E^*,\ \forall t \ge 0.$$

Thus, $E(\cdot) \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^N)$. We recall (2.6) that $\theta_c(t) = \frac{1}{N} \sum_{i=1}^N \theta_i^0$. Hence, for any $i = 1, 2, \dots, N$

$$|\theta_i(t)| \le |\theta_i(t) - \theta_c(t)| + |\theta_c(t)| \le D(\theta(t)) + |\theta_c(t)| \le \frac{\pi}{2} + \frac{1}{N} \left| \sum_{i=1}^N \theta_i^0 \right|,$$

so $\theta(\cdot) \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^N)$. □

*Proof of Theorem 2.3.* The result is a direct consequence of Lemma 2.1, Lemma 2.2 and Proposition 3.1. □

## 4. Synchronization of second-order model

In this section, we study the emergent dynamics of (2.7)-(2.9). In order to show the boundness of phases we introduce an energy functional $\mathcal{E}$ as follows:

$$\mathcal{E}[\theta, \omega] := \underbrace{\varepsilon \sum_{i=1}^N \gamma_i (\theta_i - \theta_c)^2 + 2\varepsilon \sum_{i=1}^N m_i (\theta_i - \theta_c) \omega_i + \mu \sum_{i=1}^N m_i \omega_i^2}_{\mathcal{E}_1[\theta, \omega]} + \underbrace{\mu \sum_{(i,j) \in \mathcal{W}} B_{ij} E_i E_j \left(1 - \cos(\theta_i - \theta_j)\right)}_{\mathcal{E}_2[\theta]}.$$

where $\varepsilon$ and $\mu$ are positive constants. We derive some basic properties for $\mathcal{E}[\theta, \omega]$ and some estimates on the evolution of $\mathcal{E}[\theta, \omega]$ along the flow (2.7).

**Lemma 4.1.** *Let $(\theta(t), E(t))$ be a solution of (2.7)-(2.9) satisfying the following conditions:*

*(1)* $\frac{\varepsilon}{\mu} < \frac{\gamma_l}{2m_u}$,
*(2)* $\min\limits_{1 \le i \le N} E_i^0 > 0,\ \sup\limits_{t \in [0, T^*)} D(\theta(t)) < D_0,\ D_0 \in (0, \frac{\pi}{2})$,
*(3)* $\max\limits_{1 \le i \le N} \left\{ -\beta_i + \sum_{j=1}^N B_{ij} \right\} < 0$.

*Then we have that for any $i \in \{1, 2, \dots, N\}$*

$$E_* \le E_i(t) \le E_\alpha^*,\ \forall t \in [0, T^*)$$

*and*

$$C_1 \mathcal{D}(t) \le \mathcal{E}[\theta, \omega] \le C_2 \mathcal{D}(t),\ t \in [0, T^*).$$

*Here*

$$\mathcal{D}(t) = \mathcal{D}[\theta, \omega] := \|\omega\|_2^2 + \|\theta - \theta_c \mathbb{1}\|_2^2, \quad \mathbb{1} := (1, 1, \dots, 1)^{\mathrm{T}} \in \mathbb{R}^N,$$

*and $C_1, C_2, E_*, E_\alpha^*$ are specified as in Subsection 2.4.*

*Proof.* We derive the results by two steps.

• Step 1. We derive the equivalence between $\mathcal{E}_1[\theta,\omega]$ and $\mathcal{D}(t)$. The cross term $(\theta_i-\theta_c)\omega_i$ can be estimated by Young's inequality:

$$|(\theta_i-\theta_c)\omega_i|\le \frac{\varepsilon}{\mu}(\theta_i-\theta_c)^2+\frac{\mu}{4\varepsilon}\omega_i^2.$$

Then, we have

$$-\frac{2\varepsilon^2 m_u}{\mu\gamma_l}\gamma_i(\theta_i-\theta_c)^2-\frac{\mu}{2}m_i\omega_i^2\le 2\varepsilon m_i(\theta_i-\theta_c)\omega_i\le \frac{2\varepsilon^2 m_u}{\mu\gamma_l}\gamma_i(\theta_i-\theta_c)^2+\frac{\mu}{2}m_i\omega_i^2.$$

Therefore,

$$\left(\varepsilon\gamma_l-\frac{2\varepsilon^2 m_u}{\mu}\right)\sum_{i=1}^N(\theta_i-\theta_c)^2+\frac{\mu m_l}{2}\sum_{i=1}^N\omega_i^2\le\mathcal{E}_1[\theta,\omega]\le\left(\varepsilon\gamma_u+\frac{2\varepsilon^2 m_u\gamma_u}{\mu\gamma_l}\right)\sum_{i=1}^N(\theta_i-\theta_c)^2+\frac{3\mu m_u}{2}\sum_{i=1}^N\omega_i^2,$$

i.e.,

$$\left(\varepsilon\gamma_l-\frac{2\varepsilon^2 m_u}{\mu}\right)\|\theta-\theta_c\mathbb{1}\|^2+\frac{\mu m_l}{2}\|\omega\|^2\le\mathcal{E}_1[\theta,\omega]\le\left(\varepsilon\gamma_u+\frac{2\varepsilon^2 m_u\gamma_u}{\mu\gamma_l}\right)\|\theta-\theta_c\mathbb{1}\|^2+\frac{3\mu m_u}{2}\|\omega\|^2.$$

• Step 2: We derive the equivalence between $\mathcal{E}_2[\theta]$ and $\mathcal{D}(t)$. As $\mathcal{E}_2[\theta]$ depends on the variable $E_i$, we need to derive a lower and upper bound for $E_i$. We claim that for any $i\in\{1,2,\dots,N\}$

$$E_*\le E_i(t)\le E_\alpha^*,\quad \forall t\in[0,T^*). \tag{4.1}$$

*Proof of Claim* (4.1)*:* It follows from lemma 2.3 and (2.8)-(2.9) that

$$\alpha_i\dot{E}_i(t)=q_i-\beta_iE_i(t)+\sum_{j=1}^N B_{ij}E_j(t)\cos(\theta_i(t)-\theta_j(t))\ge q_i-\beta_iE_i(t),\quad\forall t\in[0,T^*).$$

By the comparison principle, we obtain for any $i\in\{1,2,\dots,N\}$,

$$E_i(t)\ge E_*,\ \forall t\in[0,T^*). \tag{4.2}$$

On the other hand, We note that

$$\alpha_i\dot{E}_i(t)=q_i-\beta_iE_i(t)+\sum_{j=1}^N B_{ij}E_j(t)\cos(\theta_i(t)-\theta_j(t))\le q_i-\beta_iE_i(t)+\sum_{j=1}^N B_{ij}E_j(t),\quad\forall t\in[0,T^*).$$

Hence,

$$\begin{aligned}\sum_{i=1}^N\alpha_i\dot{E}_i(t)&\le\sum_{i=1}^N q_i-\sum_{i=1}^N\beta_iE_i(t)+\sum_{i=1}^N\sum_{j=1}^N B_{ij}E_j(t)\\&=\sum_{i=1}^N q_i+\sum_{i=1}^N\left(-\beta_i+\sum_{i=1}^N B_{ij}\right)E_i(t)\\&\le\sum_{i=1}^N q_i-K_\alpha\sum_{i=1}^N\alpha_iE_i(t),\quad\forall t\in[0,T^*).\end{aligned}$$

By the comparison principle, we have

$$\sum_{i=1}^{N} \alpha_i E_i(t) \le \max\left\{ \frac{\sum_{i=1}^{N} q_i}{K_\alpha}, \sum_{i=1}^{N} \alpha_i E_i^0 \right\}, \quad \forall t \in [0, T^*),$$

then for any $i \in \{1, 2, \dots, N\}$,

$$E_i(t) \le \max\left\{ \frac{\sum_{i=1}^{N} q_i}{\alpha_l K_\alpha}, \frac{\sum_{i=1}^{N} \alpha_i E_i^0}{\alpha_l} \right\} = E_\alpha^*, \quad \forall t \in [0, T^*). \tag{4.3}$$

We combine (4.2) and (4.3) to obtain the estimate in (4.1).

We now derive an estimate for $\mathcal{E}_2[\theta]$. Since $\theta_i - \theta_j \in [-D_0, D_0]$, we have

$$\frac{1 - \cos D_0}{D_0^2} |\theta_i(t) - \theta_j(t)|^2 \le 1 - \cos(\theta_i(t) - \theta_j(t)) \le \frac{1}{2} |\theta_i(t) - \theta_j(t)|^2, \quad \forall t \in [0, T^*).$$

Here, the left inequality relies on the fact that $y \to \frac{1 - \cos y}{y^2}$ is an even function which is monotonically decreasing on $(0, \pi)$. Therefore, we can derive

$$\begin{aligned}
\mathcal{E}_2[\theta](t) &= \mu \sum_{(i,j) \in \mathcal{W}} B_{ij} E_i(t) E_j(t) \left(1 - \cos(\theta_i(t) - \theta_j(t))\right) \\
&\le \mu B_u E_\alpha^{*2} \sum_{(i,j) \in \mathcal{W}} \left(1 - \cos(\theta_i(t) - \theta_j(t))\right) \\
&\le \frac{\mu B_u E_\alpha^{*2}}{2} \sum_{(i,j) \in \mathcal{W}} |\theta_i(t) - \theta_j(t)|^2 \\
&\le \frac{\mu B_u E_\alpha^{*2}}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} |\theta_i(t) - \theta_j(t)|^2 \\
&= \mu N B_u E_\alpha^{*2} \|\theta(t) - \theta_c \mathbb{1}(t)\|^2, \quad t \in [0, T^*),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_2[\theta](t) &= \mu \sum_{(i,j) \in \mathcal{W}} B_{ij} E_i(t) E_j(t) \left(1 - \cos(\theta_i(t) - \theta_j(t))\right) \\
&\ge \mu B_l E_*^2 \sum_{(i,j) \in \mathcal{W}} \left(1 - \cos(\theta_i(t) - \theta_j(t))\right) \\
&\ge \frac{\mu B_l E_*^2 (1 - \cos D_0)}{D_0^2} \sum_{(i,j) \in \mathcal{W}} |\theta_i(t) - \theta_j(t)|^2 \\
&\ge \frac{\mu B_l E_*^2 L_* (1 - \cos D_0)}{D_0^2} \sum_{i=1}^{N} \sum_{j=1}^{N} |\theta_i(t) - \theta_j(t)|^2 \\
&= \frac{2\mu N B_l E_*^2 L_* (1 - \cos D_0)}{D_0^2} \|\theta(t) - \theta_c \mathbb{1}(t)\|^2, \quad t \in [0, T^*).
\end{aligned}$$

Here we used Lemma 2.4 and the relation

$$\sum_{i=1}^{N} \sum_{j=1}^{N} |\theta_i(t) - \theta_j(t)|^2 = 2N \|\theta(t) - \theta_c \mathbb{1}(t)\|^2. \tag{4.4}$$

Finally, we combine Step1 and Step2 to obtain

$$C_1\mathcal{D}(t) \le \mathcal{E}[\theta,\omega] \le C_2\mathcal{D}(t),\ t \in [0,T^*).$$

□

We recall that the system (2.7) can be rewritten as

$$\begin{aligned}
&\dot\theta_i = \omega_i, \quad \sum_{i=1}^N \Omega_i = 0,\\
&m_i\dot\omega_i = -\gamma_i\omega_i + \Omega_i + \sum_{j=1}^N B_{ij}E_iE_j\sin(\theta_j-\theta_i),\\
&\alpha_i\dot E_i = q_i - \beta_iE_i + \sum_{j=1}^N B_{ij}E_j\cos(\theta_j-\theta_i),
\end{aligned}\tag{4.5}$$

For notational simplicity, let's denote

$$\mathcal{E}(t) := \mathcal{E}[\theta(t),\omega(t)],$$

where $(\theta(t),\omega(t))$ is a solution to the system (2.7) or (4.5). Next we will derive a differential inequality for the virtual energy $\mathcal{E}(t)$. We begin with two lemmas.

**Lemma 4.2.** *Let $(\theta(t),E(t))$ be a solution of (2.7)-(2.9) satisfying the following conditions:*

*(1)* $\frac{\varepsilon}{\mu} < \frac{\gamma_l}{2m_u}$,

*(2)* $\min\limits_{1\le i\le N} E_i^0 > 0,\ \sup\limits_{t\in[0,T^*)} D(\theta(t)) < D_0,\ D_0 \in (0,\frac{\pi}{2})$,

*(3)* $\max\limits_{1\le i\le N}\left\{-\beta_i + \sum_{j=1}^N B_{ij}\right\} < 0.$

*Then we have the following relation:*

$$\sum_{(i,j)\in\mathcal{W}} B_{ij}E_i(t)E_j(t)\sin(\theta_j(t)-\theta_i(t))(\theta_j(t)-\theta_i(t)) \ge \frac{2NB_lL_*E_*^2\sin D_0}{D_0}\|\theta(t)-\theta_c\mathbb{1}(t)\|^2,\ \forall t\in[0,T^*).$$

*Proof.* Using Lemma 4.1, Lemma 2.4, (2.8), (2.9), (4.4) and

$$(\theta_j(t)-\theta_i(t))\sin(\theta_j(t)-\theta_i(t)) \ge \frac{\sin D_0}{D_0}(\theta_j(t)-\theta_i(t))^2,\ \forall t\in[0,T^*),$$

we obtain

$$\begin{aligned}
&\sum_{(i,j)\in\mathcal{W}} B_{ij}E_i(t)E_j(t)\sin(\theta_j(t)-\theta_i(t))(\theta_j(t)-\theta_i(t)) \ge B_lE_*^2\sum_{(i,j)\in\mathcal{W}}\sin(\theta_j(t)-\theta_i(t))(\theta_j(t)-\theta_i(t))\\
&\quad\ge \frac{B_lE_*^2\sin D_0}{D_0}\sum_{(i,j)\in\mathcal{W}}|\theta_j(t)-\theta_i(t)|^2 \ge \frac{B_lE_*^2L_*\sin D_0}{D_0}\sum_{i=1}^N\sum_{j=1}^N|\theta_j(t)-\theta_i(t)|^2\\
&\quad= \frac{2NB_lL_*E_*^2\sin D_0}{D_0}\|\theta(t)-\theta_c\mathbb{1}(t)\|^2, \quad \forall t\in[0,T^*).
\end{aligned}$$

□

**Lemma 4.3.** *Let $(\theta(t),E(t))$ be a solution to the coupled system (2.7)-(2.9), then*

*(1)* $\dot\theta_s + \dot\omega_s = 0$,

*(2)* $\theta_s = \sum_{i=1}^N \gamma_i(\theta_i - \theta_c) + \mathrm{tr}(\Gamma)\theta_c$.

*where* $\theta_s := \sum_{i=1}^N \gamma_i\theta_i,\ \omega_s := \sum_{i=1}^N m_i\omega_i,\ \Gamma := \mathrm{diag}\{\gamma_1, \gamma_2, \dots, \gamma_N\}$.

*Proof.*

$$\dot\theta_s + \dot\omega_s = \sum_{i=1}^N \gamma_i\dot\theta_i + \sum_{i=1}^N m_i\ddot\theta_i = \sum_{i=1}^N \Omega_i - \sum_{i=1}^N\sum_{j=1}^N B_{ij}E_iE_j\sin(\theta_i - \theta_j) = 0,$$

and

$$\sum_{i=1}^N \gamma_i(\theta_i - \theta_c) + \mathrm{tr}(\Gamma)\theta_c = \sum_{i=1}^N \gamma_i\theta_i - \theta_c\mathrm{tr}(\Gamma) + \mathrm{tr}(\Gamma)\theta_c = \theta_s.$$

Here we used the restriction $\sum_{i=1}^N \Omega_i = 0$. □

**Proposition 4.1.** *Let* $(\theta(t), E(t))$ *be a solution of* (2.7)-(2.9) *satisfying the following conditions:*

*(1)* $\frac{\varepsilon}{\mu} \in \left(\frac{2\kappa N E_\alpha^* B_u D_0}{2NB_lL_*E_*^2\sin D_0 - \lambda D_0}, \frac{\gamma_l}{2m_u+\lambda}\right)$,

*(2)* $\min\limits_{1\le i\le N} E_i^0 > 0,\ \sup\limits_{t\in[0,T^*)} D(\theta(t)) < D_0,\ D_0 \in (0, \frac{\pi}{2})$,

*(3)* $\max\limits_{1\le i\le N}\left\{-\beta_i + \sum_{j=1}^N B_{ij}\right\} < 0$.

*Then we have*

$$\frac{d}{dt}\mathcal{E}(t) + \frac{C_L}{C_2}\mathcal{E}(t) \le \frac{2\sqrt{2}\max\{\varepsilon, \mu\}\|\Omega\|}{\sqrt{C_1}}\sqrt{\mathcal{E}(t)},\ t \in [0, T^*),$$

*where* $C_1, C_2, C_L$ *and* $\lambda$ *are given as in Subsection 2.4.2.*

*Proof.* By the definition of $\mathcal{E}(t)$, we observe that

$$\mathcal{E}(t) = \underbrace{\left(\varepsilon\sum_{i=1}^N \gamma_i\theta_i^2 + 2\varepsilon\sum_{i=1}^N m_i\theta_i\omega_i + \mu\sum_{i=1}^N m_i\omega_i^2 + \mathcal{E}_2[\theta]\right)}_{I(t)} + \underbrace{\left(-2\varepsilon\theta_s\theta_c + \varepsilon\mathrm{tr}(\Gamma)\theta_c^2 - 2\varepsilon\omega_s\theta_c\right)}_{J(t)}.$$

Therefore, we derive the differential inequality by two steps.

• Step 1. We first estimate $\frac{d}{dt}I(t)$. This is divided into there parts. First, we multiply $2\theta_i$ on both sides of the second equation in (4.5)

$$2m_i\theta_i\dot\omega_i = -2\gamma_i\theta_i\omega_i + 2\Omega_i\theta_i + 2\sum_{j=1}^N B_{ij}E_iE_j\sin(\theta_j - \theta_i)\theta_i.$$

Sum it over $i$, and then use the symmetry to obtain

$$\begin{aligned}2\sum_{i=1}^N m_i\theta_i\dot\omega_i &= -2\sum_{i=1}^N \gamma_i\theta_i\omega_i + 2\sum_{i=1}^N \Omega_i\theta_i + 2\sum_{(i,j)\in\mathcal{W}} B_{ij}E_iE_j\sin(\theta_j - \theta_i)\theta_i \\ &= -\frac{d}{dt}\sum_{i=1}^N \gamma_i\theta_i^2 + 2\sum_{i=1}^N \Omega_i\theta_i - \sum_{(i,j)\in\mathcal{W}} B_{ij}E_iE_j\sin(\theta_j - \theta_i)(\theta_j - \theta_i)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{d}{dt}\sum_{i=1}^N \gamma_i\theta_i^2 + 2\sum_{i=1}^N \Omega_i(\theta_i-\theta_c) - \sum_{(i,j)\in\mathcal{W}} B_{ij}E_iE_j\sin(\theta_j-\theta_i)(\theta_j-\theta_i)\\
&\le -\frac{d}{dt}\sum_{i=1}^N \gamma_i\theta_i^2 + 2\|\Omega\|\|\theta-\theta_c\mathbb{1}\| - \frac{2NB_lL_*E_*^2\sin D_0}{D_0}\|\theta-\theta_c\mathbb{1}\|^2,
\end{aligned}
$$

here, we used $\sum_{i=1}^N \Omega_i = 0$ and Lemma 4.2. Because of $\theta_i\dot\omega_i = \frac{d}{dt}(\omega_i\theta_i) - \omega_i^2$, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(2\sum_{i=1}^N m_i\omega_i\theta_i + \sum_{i=1}^N \gamma_i\theta_i^2\right)\\
&\le 2\sum_{i=1}^N m_i\omega_i^2 + 2\|\Omega\|\|\theta-\theta_c\mathbb{1}\| - \frac{2NB_lL_*E_*^2\sin D_0}{D_0}\|\theta-\theta_c\mathbb{1}\|^2\\
&\le 2m_u\|\omega\|^2 + 2\|\Omega\|\|\theta-\theta_c\mathbb{1}\| - \frac{2NB_lL_*E_*^2\sin D_0}{D_0}\|\theta-\theta_c\mathbb{1}\|^2.
\end{aligned}
\tag{4.6}
$$

Second, we multiply $2\omega_i$ on both sides of the second equation in (4.5)

$$
2m_i\omega_i\dot\omega_i = -2\gamma_i\omega_i^2 + 2\Omega_i\omega_i + 2\sum_{j=1}^N B_{ij}E_iE_j\sin(\theta_j-\theta_i)\omega_i.
$$

Sum it over $i$, and then use the symmetry to obtain

$$
\begin{aligned}
\frac{d}{dt}\sum_{i=1}^N m_i\omega_i^2 &= -2\sum_{i=1}^N \gamma_i\omega_i^2 + 2\sum_{i=1}^N \Omega_i\omega_i + 2\sum_{i=1}^N\sum_{j=1}^N B_{ij}E_iE_j\sin(\theta_j-\theta_i)\omega_i\\
&= -2\sum_{i=1}^N \gamma_i\omega_i^2 + 2\sum_{i=1}^N \Omega_i\omega_i - \sum_{(i,j)\in\mathcal{W}} B_{ij}E_iE_j\sin(\theta_j-\theta_i)(\omega_j-\omega_i).
\end{aligned}
\tag{4.7}
$$

Thirdly, we estimate the term $\frac{d}{dt}\mathcal{E}_2[\theta]$. It follows from Lemma 4.1 that

$$
\alpha_i\dot E_i = q_i - \beta_iE_i + \sum_{j=1}^N B_{ij}E_j\cos(\theta_i-\theta_j) \le \max_{1\le i\le N}\{q_i-\beta_iE_*\} + NB_uE_\alpha^*,\ t\in[0,T^*),
$$

which implies

$$
\dot E_i \le \kappa,\ t\in[0,T^*).
$$

Then we obtain

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}_2[\theta] &= \frac{d}{dt}\left(\mu\sum_{(i,j)\in\mathcal{W}} B_{ij}E_iE_j\left(1-\cos(\theta_i-\theta_j)\right)\right)\\
&= 2\mu\sum_{(i,j)\in\mathcal{W}} B_{ij}\dot E_iE_j\left(1-\cos(\theta_i-\theta_j)\right) + \mu\sum_{(i,j)\in\mathcal{W}} B_{ij}E_iE_j\sin(\theta_i-\theta_j)(\omega_i-\omega_j)\\
&\le 2\kappa\mu E_\alpha^*B_u\sum_{(i,j)\in\mathcal{W}}\left(1-\cos(\theta_i-\theta_j)\right) + \mu\sum_{(i,j)\in\mathcal{W}} B_{ij}E_iE_j\sin(\theta_i-\theta_j)(\omega_i-\omega_j)\\
&\le \mu\kappa E_\alpha^*B_u\sum_{(i,j)\in\mathcal{W}}|\theta_i-\theta_j|^2 + \mu\sum_{(i,j)\in\mathcal{W}} B_{ij}E_iE_j\sin(\theta_i-\theta_j)(\omega_i-\omega_j)
\end{aligned}
$$

$$\leq \mu\kappa E_\alpha^* B_u \sum_{i=1}^N \sum_{j=1}^N |\theta_i - \theta_j|^2 + \mu \sum_{(i,j)\in\mathcal{W}} B_{ij} E_i E_j \sin(\theta_i - \theta_j)(\omega_i - \omega_j).$$

Therefore, we have

$$\frac{d}{dt}\mathcal{E}_2[\theta] \leq 2\mu\kappa N E_\alpha^* B_u \|\theta - \theta_c \mathbb{1}\|^2 + \mu \sum_{(i,j)\in\mathcal{W}} B_{ij} E_i E_j \sin(\theta_i - \theta_j)(\omega_i - \omega_j). \tag{4.8}$$

We now combine (4.6)-(4.8) to obtain that

$$\begin{aligned}
\frac{d}{dt} I(t) &\leq 2\varepsilon m_u \|\omega\|^2 + 2\varepsilon \|\Omega\| \|\theta - \theta_c \mathbb{1}\| - \frac{2\varepsilon N B_l L_* E_*^2 \sin D_0}{D_0} \|\theta - \theta_c \mathbb{1}\|^2 \\
&\quad - 2\mu \sum_{i=1}^N \gamma_i \omega_i^2 + 2\mu \sum_{i=1}^N \Omega_i \omega_i - \mu \sum_{(i,j)\in\mathcal{W}} B_{ij} E_i E_j \sin(\theta_i - \theta_j)(\omega_i - \omega_j) \\
&\quad + 2\kappa\mu N E_\alpha^* B_u \|\theta - \theta_c \mathbb{1}\|^2 + \mu \sum_{(i,j)\in\mathcal{W}} B_{ij} E_i E_j \sin(\theta_i - \theta_j)(\omega_i - \omega_j) \\
&\leq 2\varepsilon \|\Omega\| \|\theta - \theta_c \mathbb{1}\| + 2\varepsilon m_u \|\omega\|^2 - 2\mu\gamma_l \|\omega\|^2 + 2\mu \|\Omega\| \|\omega\| \\
&\quad - \left( \frac{2\varepsilon N B_l L_* E_*^2 \sin D_0}{D_0} - 2\kappa\mu N E_\alpha^* B_u \right) \|\theta - \theta_c \mathbb{1}\|^2, \ t \in [0, T^*).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\frac{d}{dt} I(t) + 2(\mu\gamma_l - \varepsilon m_u)\|\omega\|^2 + \left( \frac{2\varepsilon N B_l L_* E_*^2 \sin D_0}{D_0} - 2\kappa\mu N E_\alpha^* B_u \right) \|\theta - \theta_c \mathbb{1}\|^2 \\
&\leq 2\max\{\varepsilon, \mu\} \|\Omega\| (\|\omega\| + \|\theta - \theta_c \mathbb{1}\|) \leq 2\sqrt{2} \max\{\varepsilon, \mu\} \|\Omega\| \sqrt{\|\theta - \theta_c \mathbb{1}\|^2 + \|\omega\|^2} \\
&= 2\sqrt{2} \max\{\varepsilon, \mu\} \|\Omega\| \sqrt{\mathcal{D}(t)} \leq \frac{2\sqrt{2}\max\{\varepsilon, \mu\}\|\Omega\|}{\sqrt{C_1}} \sqrt{\mathcal{E}(t)}, \ t \in [0, T^*).
\end{aligned}$$

Hence,

$$\frac{d}{dt} I(t) + C_3 \mathcal{D}(t) \leq \frac{2\sqrt{2}\max\{\varepsilon, \mu\}\|\Omega\|}{\sqrt{C_1}} \sqrt{\mathcal{E}(t)}, \ t \in [0, T^*),$$

where $C_3 := \min\left\{ 2(\mu\gamma_l - \varepsilon m_u), \frac{2\varepsilon N B_l L_* E_*^2 \sin D_0}{D_0} - 2\kappa\mu N E_\alpha^* B_u \right\}$.

• Step 2. We now estimate $\frac{d}{dt} J(t)$. Let $\omega_c := \frac{1}{N}\sum_{i=1}^N \omega_i$ and we use Lemma 4.3 to obtain

$$\begin{aligned}
J &= -2\varepsilon \left( \dot{\theta}_s \theta_c + \theta_s \dot{\theta}_c \right) + 2\varepsilon \mathrm{tr}(\Gamma) \theta_c \dot{\theta}_c - 2\varepsilon \left( \dot{\omega}_s \theta_c + \omega_s \dot{\theta}_c \right) \\
&= 2\varepsilon \left( -\theta_s \dot{\theta}_c + \mathrm{tr}(\Gamma) \theta_c \dot{\theta}_c - \omega_s \dot{\theta}_c \right) \\
&= -2\varepsilon \dot{\theta}_c \sum_{i=1}^N \gamma_i (\theta_i - \theta_c) - 2\varepsilon \omega_s \dot{\theta}_c \\
&= -2\varepsilon \omega_c \sum_{i=1}^N \gamma_i (\theta_i - \theta_c) - 2\varepsilon \omega_s \omega_c.
\end{aligned}$$

Note that

$$\sum_{i=1}^N \gamma_i(\theta_i-\theta_c)=\sum_{i=1}^N \hat{\gamma}_i(\theta_i-\theta_c) \quad and \quad \omega_s=\sum_{i=1}^N \hat{m}_i\omega_i+\mathrm{tr}(M_1)\omega_c,$$

where $M_1 := \mathrm{diag}\{m_1, m_2, \dots, m_N\}$. This yields

$$\begin{aligned}
J &= -2\varepsilon\omega_c\sum_{i=1}^N \hat{\gamma}_i(\theta_i-\theta_c)-2\varepsilon\left(\sum_{i=1}^N \hat{m}_i\omega_i+\mathrm{tr}(M_1)\omega_c\right)\omega_c \\
&\le -2\varepsilon\omega_c\sum_{i=1}^N \hat{\gamma}_i(\theta_i-\theta_c)-2\varepsilon\omega_c\sum_{i=1}^N \hat{m}_i\omega_i \\
&\le \frac{2\varepsilon}{N}\Big|\sum_{i=1}^N\omega_i\Big|\Big|\sum_{i=1}^N \hat{\gamma}_i(\theta_i-\theta_c)\Big|+\frac{2\varepsilon}{N}\Big|\sum_{i=1}^N\omega_i\Big|\Big|\sum_{i=1}^N \hat{m}_i\omega_i\Big| \\
&\le \frac{2\varepsilon}{N}\sqrt{N}\|\omega\|\sqrt{\mathrm{tr}(\hat{\Gamma}^2)}\|\theta-\theta_c\mathbb{1}\|+\frac{2\varepsilon}{N}\sqrt{N}\|\omega\|\sqrt{\mathrm{tr}(\hat{M}^2)}\|\omega\| \\
&= \frac{2\varepsilon}{\sqrt{N}}\sqrt{\mathrm{tr}(\hat{\Gamma}^2)}\|\omega\|\|\theta-\theta_c\mathbb{1}\|+\frac{2\varepsilon}{\sqrt{N}}\sqrt{\mathrm{tr}(\hat{M}^2)}\|\omega\|^2 \\
&\le \frac{\varepsilon\sqrt{\mathrm{tr}(\hat{\Gamma}^2)}}{\sqrt{N}}(\|\omega\|^2+\|\theta-\theta_c\mathbb{1}\|^2)+\frac{2\varepsilon\sqrt{\mathrm{tr}(\hat{M}^2)}}{\sqrt{N}}\|\omega\|^2 \\
&\le \frac{\varepsilon\left(\sqrt{\mathrm{tr}(\hat{\Gamma}^2)}+2\sqrt{\mathrm{tr}(\hat{M}^2)}\right)}{\sqrt{N}}\mathcal{D}(t) \\
&= \varepsilon\lambda\mathcal{D}(t).
\end{aligned}$$

We now combine the above estimates in Step 1 and Step 2 to see that, for $t\in[0,T^*)$,

$$\frac{d}{dt}\mathcal{E}(t)+C_3\mathcal{D}(t)\le\varepsilon\lambda\mathcal{D}(t)+\frac{2\sqrt{2}\max\{\varepsilon,\mu\}\|\Omega\|}{\sqrt{C_1}}\sqrt{\mathcal{E}(t)}.$$

We note that $C_L=C_3-\varepsilon\lambda$ and use Lemma 4.1 to see

$$\frac{d}{dt}\mathcal{E}(t)+\frac{C_L}{C_2}\mathcal{E}(t)\le\frac{2\sqrt{2}\max\{\varepsilon,\mu\}\|\Omega\|}{\sqrt{C_1}}\sqrt{\mathcal{E}(t)},\ t\in[0,T^*).$$

□

**Proposition 4.2.** *Let* $(\theta(t),E(t))$ *be a solution to the coupled system* (2.7)-(2.9) *satisfying hypotheses* **(H1)-(H3)**. *Then we have*

$$\sup_{t\ge0}D(\theta(t))\le D_0.$$

*Proof.* For the sake of notational simplicity, we set

$$y(t):=\sqrt{\mathcal{E}(t)},\ t\ge0.$$

We define the set $\mathcal{T}$ and its supremum:

$$\mathcal{T}:=\left\{T\in[0,+\infty)\,\Big|\,y(t)<\frac{\sqrt{C_1}}{2}D_0,\ \forall\,t\in[0,T)\right\},\quad T^*:=\sup\mathcal{T}.$$

Note that by the assumption,

$$y(0) < \frac{\sqrt{C_1}}{2} D_0.$$

Due to the continuity of $y(t)$, there exists a positive constant $T > 0$ such that $T \in \mathcal{T}$. We now claim that

$$T^* = \infty.$$

Suppose the opposite, i.e., $T^*$ is finite. Then we should have

$$y(t) < \frac{\sqrt{C_1}}{2} D_0,\ t \in [0, T^*), \quad y(T^*) = \frac{\sqrt{C_1}}{2} D_0.$$

Note that on the interval $[0, T^*)$, we can derive that

$$\begin{aligned}\max_{1\le i,j\le N} |\theta_i(t) - \theta_j(t)|^2 &\le 4 \max_{1\le i,j\le N} |\theta_i(t) - \theta_c(t)|^2 \le 4 \sum_{i=1}^N |\theta_i(t) - \theta_c(t)|^2 \\ &= 4\|\theta - \theta_c \mathbb{1}\|^2 \le 4\mathcal{D}(t) \le \frac{4}{C_1}\mathcal{E}(t) = \frac{4}{C_1}(y(t))^2 \\ &< \frac{4}{C_1}\left(\frac{\sqrt{C_1}}{2} D_0\right)^2 = D_0^2.\end{aligned}$$

We use Proposition 4.1 to obtain

$$\frac{d}{dt} y(t) \le \frac{\sqrt{2}\max\{\varepsilon, \mu\}\|\Omega\|}{\sqrt{C_1}} - \frac{C_L}{2C_2} y(t),\ t \in [0, T^*).$$

By the comparison principle, we have

$$y(t) \le \max\left\{ y(0), \frac{2\sqrt{2} C_2 \max\{\varepsilon, \mu\}\|\Omega\|}{\sqrt{C_1} C_L} \right\},\ t \in [0, T^*).$$

Letting $t \to T^{*-}$ yields

$$y(T^*) \le \max\left\{ y(0), \frac{2\sqrt{2} C_2 \max\{\varepsilon, \mu\}\|\Omega\|}{\sqrt{C_1} C_L} \right\} < \frac{\sqrt{C_1}}{2} D_0,$$

which contradicts

$$y(T^*) = \frac{\sqrt{C_1}}{2} D_0,$$

i.e., $T^* = \infty$. Repeat the above steps, we have

$$\sup_{t\ge 0} D(\theta(t)) \le D_0.$$

□

**Proposition 4.3.** *Let* $(\theta(t), E(t))$ *be a solution to the coupled system* (2.7)-(2.9) *satisfying hypotheses* **(H1)-(H3)**. *Then, we have*

$$E(\cdot) \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^N), \quad \theta(\cdot) \in \ell^{1,\infty}(\mathbb{R}^+, \mathbb{R}^N).$$

*Proof.* It follows from Lemma 4.1 and Proposition 4.2 that for any $i \in \{1, 2, \dots, N\}$

$$E_* \le E_i(t) \le E^*_\alpha, \quad \forall t \in [0, \infty).$$

Thus, $E(\cdot) \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^N)$. For $\omega_i := \dot{\theta}_i$, we have

$$m_i \frac{d}{dt}\omega_i + \gamma_i \omega_i = \Omega_i - \sum_{j=1}^N B_{ij} E_i E_j \sin(\theta_i - \theta_j).$$

Note that $\omega_i$ is an analytic function of $t$. This implies that the zero-set $t : \omega_i = 0$ is countable and finite in any finite time-interval, i.e., $|\omega_i|$ is piecewise differentiable and continuous. We multiply the above relation by $\mathrm{sgn}(\omega_i)$ to get

$$\frac{d|\omega_i|}{dt} + \frac{\gamma_i}{m_i}|\omega_i| \le \frac{1}{m_i}\left[|\Omega_i| + {E^*_\alpha}^2 \sum_{j=1}^N B_{ij}\right], \ a.e. t \ge 0.$$

We now use Gronwall inequality and continuity of $|\omega_i|$ to obtain that for all $t > 0$,

$$\begin{aligned} |\omega_i(t)| &\le |\omega_i(0)| e^{-\frac{\gamma_i}{m_i}t} + \frac{1}{\gamma_i}\left[|\Omega_i| + {E^*_\alpha}^2 \sum_{j=1}^N B_{ij}\right]\left(1 - e^{-\frac{\gamma_i}{m_i}t}\right) \\ &\le |\omega_i(0)| + \frac{1}{\gamma_i}\left[|\Omega_i| + {E^*_\alpha}^2 \sum_{j=1}^N B_{ij}\right], \end{aligned}$$

i.e.,

$$|\dot{\theta}_i(t)| \le |\omega_i(0)| + \frac{1}{\gamma_i}\left[|\Omega_i| + {E^*_\alpha}^2 \sum_{j=1}^N B_{ij}\right],$$

which indicates $\dot{\theta}(\cdot) \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^N)$. On the other hand, we recall Lemma 4.3 to get

$$\theta_s(t) + \omega_s(t) = \theta_s(0) + \omega_s(0), \ \forall t \ge 0.$$

This means that

$$|\theta_s(t)| \le |\theta_s(t) + \omega_s(t)| + |\omega_s(t)| \le |\theta_s(0) + \omega_s(0)| + m_u \sum_{i=1}^N |\omega_i(t)|, \forall t \ge 0.$$

We now use $\dot{\theta}(\cdot) \in \ell^\infty(\mathbb{R}^+, \mathbb{R}^N)$ to deduce

$$|\theta_s(t)| \le K_0, \forall t \ge 0, \tag{4.9}$$

for the positive constant

$$K_0 = |\theta_s(0) + \omega_s(0)| + m_u \sum_{i=1}^N |\omega_i(0)| + m_u \sum_{i=1}^N \frac{1}{\gamma_i}\left[|\Omega_i| + {E^*_\alpha}^2 \sum_{j=1}^N B_{ij}\right].$$

Combining (4.9) and Proposition 4.2 , we see that the trajectory $\theta(\cdot)$ is bounded as a function in time $t$. So $\theta(\cdot) \in \ell^{1,\infty}(\mathbb{R}^+, \mathbb{R}^N)$. □

*Proof of Theorem 2.4.* The result is a direct consequence of Lemma 2.1, Theorem 2.1 and Proposition 4.3. □

## 5. Conclusions

In this paper, we studied the synchronization of a coupled system of Kuramoto oscillators which models the power grids with dynamic voltages. Two related models are considered in this paper: a simplified model with first-order phase dynamics and the original model with second-order phase dynamics. Thanks to the gradient formulation relying on the symmetric network structure, we employed the energy method and gradient approach. We studied the asymptotic synchronization and voltage stabilization emerging from initial configurations and parameters, and our main results provide several estimates for the region of attraction of steady states. We acknowledge that the sufficient conditions are conservative, and we believe that obtaining sharp conditions is a meaningful problem. In [22], some sharp conditions were obtained for the existence of synchronized solutions to Kuramoto model on networks with constant coupling strengths. It is then an interesting problem to bridge the gap between these sharp bounds for the existence of synchronized solutions and ours for the region of attraction, even for the network with constant couplings.

## Appendix: Proof of Theorem 2.1

For the proof of Theorem 2.1, we need the Łojasiewicz gradient inequality and Barbalat lemma as follows.

**Lemma 5.1.** [27] *Let $f : \mathbb{R}^N \to \mathbb{R}$ be real-analytic. Then for any $x_0 \in \mathbb{R}^N$, there exist constants $\delta = \delta(x_0) > 0, C = C(x_0) > 0$ and $r = r(x_0) \in (0, \frac{1}{2}]$ such that*

$$|f(x) - f(x_0)|^{1-r} \leq C\|\nabla f(x)\|, \ \forall x \in B_\delta(x_0).$$

**Lemma 5.2.** [2] *Let $g : [0, \infty) \to \mathbb{R}^+$ be a uniformly continuous function such that $\lim\limits_{t\to\infty} \int_0^t g(s)ds$ exists. Then we have $\lim\limits_{t\to\infty} g(t) = 0$.*

Next we give a complete proof for Theorem 2.1.

*Proof of Theorem 2.1.* We use the quasi-gradient flow approach and finite length argument.
• Step 1: we show (2.10) is a quasi-gradient system. Note that

$$x(t) = (\theta_1(t), \dots, \theta_N(t), E_1(t), \dots, E_N(t))^{\mathrm{T}},$$

for notational simplification we omit the time-dependence and denote

$$\theta = (\theta_1, \theta_2, \dots, \theta_N)^{\mathrm{T}}, \quad E = (E_1, E_2, \dots, E_N)^{\mathrm{T}}, \quad \omega = (\dot\theta_1, \dot\theta_2, \dots, \dot\theta_N)^{\mathrm{T}},$$

and for parameters we denote

$$M_1 = \mathrm{diag}(m_1, m_2, \dots, m_N), \quad D_1 = \mathrm{diag}(\gamma_1, \gamma_2, \dots, \gamma_N), \quad D_2 = \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_N).$$

Then the system (2.10) can be reformulated as a first-order system

$$\begin{pmatrix} \dot\theta \\ \dot\omega \\ \dot E \end{pmatrix} + F(\theta, \omega, E) = 0 \tag{5.1}$$

where

$$F(\theta, \omega, E) = \begin{pmatrix} -\omega \\ M_1^{-1} D_1 \omega + M_1^{-1} \nabla_\theta V(\theta, E) \\ D_2^{-1} \nabla_E V(\theta, E) \end{pmatrix}.$$

Let

$$\mathcal{E}_\eta(\theta,\omega,E)=\frac{1}{2}\langle M_1\omega,\omega\rangle+V(\theta,E)+\eta\langle\nabla_\theta V,D_1\omega\rangle,\ \eta>0,$$

then

$$\nabla\mathcal{E}_\eta(\theta,\omega,E)=\begin{pmatrix}\nabla_\theta V+\eta\nabla^2_{\theta\theta}VD_1\omega\\ M_1\omega+\eta D_1\nabla_\theta V\\ \nabla_E V+\eta\nabla^2_{\theta E}VD_1\omega\end{pmatrix}.$$

We claim that (5.1) is a quasi-gradient flow with energy $\mathcal{E}_\eta(\theta,\omega,E)$ for sufficiently small $\eta$, i.e., there exists a constant $\tilde{C}>0$ such that for any $(\theta,\omega,E)\in\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N$,

$$\langle\nabla\mathcal{E}_\eta(\theta,\omega,E),F(\theta,\omega,E)\rangle\geq\tilde{C}\|\nabla\mathcal{E}_\eta(\theta,\omega,E)\|\|F(\theta,\omega,E)\|. \tag{5.2}$$

*Proof of claim* (5.2). For any $(\theta,\omega,E)\in\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N$ and sufficiently small $\eta$, we have

(5.3)

$$\begin{aligned}&\langle\nabla\mathcal{E}_\eta(\theta,\omega,E),F(\theta,\omega,E)\rangle\\
=&\langle\nabla_\theta V,-\omega\rangle+\eta\langle\nabla^2_{\theta\theta}VD_1\omega,-\omega\rangle+\langle M_1\omega,M_1^{-1}D_1\omega\rangle+\langle M_1\omega,M_1^{-1}\nabla_\theta V\rangle+\langle\nabla_E V,D_2^{-1}\nabla_E V\rangle\\
&+\eta\langle D_1\nabla_\theta V,M_1^{-1}D_1\omega\rangle+\eta\langle D_1\nabla_\theta V,M_1^{-1}\nabla_\theta V\rangle+\eta\langle\nabla^2_{\theta E}VD_1\omega,D_2^{-1}\nabla_E V\rangle\\
=&-\eta\langle\nabla^2_{\theta\theta}VD_1\omega,\omega\rangle+\|\sqrt{D_1}\omega\|^2+\|\sqrt{D_2^{-1}}\nabla_E V\|^2+\eta\langle D_1\nabla_\theta V,M_1^{-1}D_1\omega\rangle\\
&+\|\sqrt{M_1^{-1}D_1}\nabla_\theta V\|^2+\eta\langle\nabla^2_{\theta E}VD_1\omega,D_2^{-1}\nabla_E V\rangle\\
\geq&\|\sqrt{D_1}\omega\|^2+\|\sqrt{D_2^{-1}}\nabla_E V\|^2+\|\sqrt{M_1^{-1}D_1}\nabla_\theta V\|^2-\eta\|\nabla^2_{\theta\theta}VD_1\omega\|\|\omega\|\\
&-\eta\|D_1\nabla_\theta V\|\|M_1^{-1}D_1\omega\|-\eta\|\nabla^2_{\theta E}VD_1\omega\|\|D_2^{-1}\nabla_E V\|\\
\geq&\tilde{C}\left(\|\omega\|^2+\|\nabla_E V\|^2+\|\nabla_\theta V\|^2\right).\end{aligned}$$

On the other hand, we can find a constant $\tilde{C}_2>0$ such that

$$\begin{aligned}\|F(\theta,\omega,E)\|&=\left(\|\omega\|^2+\|M_1^{-1}D_1\omega+M_1^{-1}\nabla_\theta V\|^2+\|D_2^{-1}\nabla_E V\|^2\right)^{\frac{1}{2}}\\
&\leq\tilde{C}_2\left(\|\omega\|+\|\nabla_E V\|+\|\nabla_\theta V\|\right).\end{aligned}\tag{5.4}$$

For $\nabla\mathcal{E}_\eta(\theta,\omega,E)$ with sufficiently small $\eta$, we can find some constant $\tilde{C}_3>0$ such that

$$\begin{aligned}&\|\nabla\mathcal{E}_\eta(\theta,\omega,E)\|\\
&=\left(\|\nabla_\theta V+\eta\nabla^2_{\theta\theta}VD_1\omega\|^2+\|M_1\omega+\eta D_1\nabla_\theta V\|^2+\|\nabla_E V+\eta\nabla^2_{\theta E}VD_1\omega\|^2\right)^{\frac{1}{2}}\\
&\leq\tilde{C}_3\left(\|\omega\|+\|\nabla_E V\|+\|\nabla_\theta V\|\right).\end{aligned}\tag{5.5}$$

The relations (5.4) and (5.5) imply that

$$\begin{aligned}\|\nabla\mathcal{E}_\eta(\theta,\omega,E)\|\|F(\theta,\omega,E)\|&\leq\tilde{C}_4\left(\|\omega\|+\|\nabla_E V\|+\|\nabla_\theta V\|\right)^2\\
&\leq\tilde{C}_5\left(\|\omega\|^2+\|\nabla_E V\|^2+\|\nabla_\theta V\|^2\right).\end{aligned}\tag{5.6}$$

We combine (5.3) and (5.6) to obtain the desired estimate in (5.2).

• Step 2: a bounded trajectory has a finite length and converges. Note that for any $(\theta,\omega,E)\in\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N$, we have

$$\frac{d}{dt}\mathcal{E}_\eta(\theta,\omega,E)=-\langle\nabla\mathcal{E}_\eta(\theta,\omega,E),F(\theta,\omega,E)\rangle\leq-\tilde{C}\|\nabla\mathcal{E}_\eta(\theta,\omega,E)\|\|F(\theta,\omega,E)\|.$$

This implies that $\mathcal{E}_\eta(\theta,\omega,E)$ is nonincreasing along the trajectory of (2.10). On the other hand, the assumption (2.11) tells that the trajectory $\{\theta(t),\omega(t),E(t)\}$ is bounded, so the range of $\mathcal{E}_\eta(\theta,\omega,E)$ is also bounded. Thus, there exists a unique limit $\mathcal{E}_\infty \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \mathcal{E}_\eta(\theta,\omega,E) = \mathcal{E}_\infty.$$

According to the Bolzano-Weierstrass theorem, there exists a sequence $\{t_n\}_{n=1}^\infty$ and $(\theta_\infty,\omega_\infty,E_\infty) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ such that

$$t_n \nearrow +\infty, \quad \lim_{t\to\infty} (\theta(t_n),\omega(t_n),E(t_n)) = (\theta_\infty,\omega_\infty,E_\infty). \tag{5.7}$$

Therefore, $\mathcal{E}_\eta(\theta_\infty,\omega_\infty,E_\infty) = \mathcal{E}_\infty$. Without loss of any generality, we may assume that $\mathcal{E}_\infty = 0$. As $\mathcal{E}_\eta(\theta,\omega,E)$ is an analytic function of $(\theta,\omega,E)$, Lemma 5.1 tells that there exist $\check{C},\delta > 0$ and $r \in (0,\frac{1}{2}]$ such that

$$\mathcal{E}_\eta^{1-r}(\theta,\omega,E) \le \check{C}\|\nabla\mathcal{E}_\eta(\theta,\omega,E)\|, \quad \forall\,(\theta,\omega,E) \in B_\delta(\theta_\infty)\times B_\delta(\omega_\infty)\times B_\delta(E_\infty). \tag{5.8}$$

We now set

$$h(t) = \mathcal{E}_\eta^r(\theta(t),\omega(t),E(t)). \tag{5.9}$$

Then $h$ is decreasing and converge to $\mathcal{E}_\infty^r$ as $t \to \infty$. Therefore, for any small $\varepsilon \in (0,\delta)$, there exists $T_0 > 0$ such that

$$0 \le h(T_0) - h(t) \le \frac{\varepsilon r \tilde{C}}{3\check{C}}, \ \forall t \ge T_0. \tag{5.10}$$

Because of (5.7), we can select $T_0$ to satisfy

$$\left\| \begin{pmatrix} \theta(T_0) \\ \omega(T_0) \\ E(T_0) \end{pmatrix} - \begin{pmatrix} \theta_\infty \\ \omega_\infty \\ E_\infty \end{pmatrix} \right\| \le \frac{\varepsilon}{3}. \tag{5.11}$$

Next, we claim

$$\begin{pmatrix} \theta(t) \\ \omega(t) \\ E(t) \end{pmatrix} \in B_\varepsilon\left(\begin{pmatrix} \theta_\infty \\ \omega_\infty \\ E_\infty \end{pmatrix}\right), \ t \ge T_0. \tag{5.12}$$

*Proof of claim* (5.12). Suppose that the opposite is true, i.e., there exists a finite time $\bar{t} > T_0$ such that

$$\begin{pmatrix} \theta(\bar{t}) \\ \omega(\bar{t}) \\ E(\bar{t}) \end{pmatrix} \notin B_\varepsilon\left(\begin{pmatrix} \theta_\infty \\ \omega_\infty \\ E_\infty \end{pmatrix}\right) \tag{5.13}$$

Let $T_1$ be the first exit time from the region

$$T_1 := \inf\left\{ t \in \mathbb{R}^+ \middle| t \ge T_0, \begin{pmatrix} \theta(t) \\ \omega(t) \\ E(t) \end{pmatrix} \notin B_\varepsilon\left(\begin{pmatrix} \theta_\infty \\ \omega_\infty \\ E_\infty \end{pmatrix}\right) \right\}.$$

The definition of $T_1$ together with (5.13) imply that

$$T_1 < \infty, \quad \begin{pmatrix} \theta(T_1) \\ \omega(T_1) \\ E(T_1) \end{pmatrix} \in \partial B_\varepsilon\left(\begin{pmatrix} \theta_\infty \\ \omega_\infty \\ E_\infty \end{pmatrix}\right) \tag{5.14}$$

and

$$\left\| \begin{pmatrix} \theta(t) \\ \omega(t) \\ E(t) \end{pmatrix} - \begin{pmatrix} \theta_\infty \\ \omega_\infty \\ E_\infty \end{pmatrix} \right\| \le \varepsilon < \delta, i.e., \begin{pmatrix} \theta(t) \\ \omega(t) \\ E(t) \end{pmatrix} \in B_\delta\left( \begin{pmatrix} \theta_\infty \\ \omega_\infty \\ E_\infty \end{pmatrix} \right), \ t \in [T_0, T_1]. \tag{5.15}$$

We now differentiate (5.9) and use (5.2), (5.8) and (5.15) to obtain

$$\begin{aligned} \frac{d}{dt}h(t) &= r\mathcal{E}_\eta^{r-1}(\theta,\omega,E)\frac{d}{dt}\mathcal{E}_\eta(\theta,\omega,E) \\ &= r\mathcal{E}_\eta^{r-1}(\theta,\omega,E)\langle \nabla\mathcal{E}_\eta(\theta,\omega,E), -F(\theta,\omega,E)\rangle \\ &\le -r\tilde{C}\mathcal{E}_\eta^{r-1}(\theta,\omega,E)\|\nabla\mathcal{E}_\eta(\theta,\omega,E)\|\|F(\theta,\omega,E)\| \\ &\le -\frac{r\tilde{C}}{\check{C}}\|F(\theta,\omega,E)\|, \quad \forall\, t \in [T_0, T_1]. \end{aligned}$$

Then

$$\int_{T_0}^t \|F(\theta,\omega,E)\|d\tau \le -\frac{\check{C}}{r\tilde{C}}(h(t) - h(T_0)), \ t \in [T_0, T_1]. \tag{5.16}$$

We combine (5.10) and (5.16) to get

$$\int_{T_0}^t \left\| \begin{pmatrix} \dot\theta(\tau) \\ \dot\omega(\tau) \\ \dot E(\tau) \end{pmatrix} \right\| d\tau = \int_{T_0}^t \|F(\theta,\omega,E)\|d\tau \le -\frac{\check{C}}{r\tilde{C}}(h(t) - h(T_0)) \le \frac{\varepsilon}{3}, \ t \in [T_0, T_1]. \tag{5.17}$$

We now use (5.11) and (5.17) to obtain

$$\begin{aligned} \left\| \begin{pmatrix} \theta(T_1) \\ \omega(T_1) \\ E(T_1) \end{pmatrix} - \begin{pmatrix} \theta_\infty \\ \omega_\infty \\ E_\infty \end{pmatrix} \right\| &\le \left\| \begin{pmatrix} \theta(T_1) \\ \omega(T_1) \\ E(T_1) \end{pmatrix} - \begin{pmatrix} \theta(T_0) \\ \omega(T_0) \\ E(T_0) \end{pmatrix} \right\| + \left\| \begin{pmatrix} \theta(T_0) \\ \omega(T_0) \\ E(T_0) \end{pmatrix} - \begin{pmatrix} \theta_\infty \\ \omega_\infty \\ E_\infty \end{pmatrix} \right\| \\ &\le \int_{T_0}^{T_1} \left\| \begin{pmatrix} \dot\theta(\tau) \\ \dot\omega(\tau) \\ \dot E(\tau) \end{pmatrix} \right\| d\tau + \frac{\varepsilon}{3} \le \frac{2\varepsilon}{3}. \end{aligned}$$

This contradicts (5.14). Therefore we have $T_1 = \infty$. This proves claim (5.12). By using the same argument as above, we see that the statement (5.16) is true for all $t \in [T_0, \infty)$. We take $t \to \infty$ in (5.17) to get

$$\int_{T_0}^\infty \left\| \begin{pmatrix} \dot\theta(\tau) \\ \dot\omega(\tau) \\ \dot E(\tau) \end{pmatrix} \right\| d\tau \le \frac{\varepsilon}{3}, \tag{5.18}$$

which means that the trajectory $\{\theta(t), \omega(t), E(t)\}$ has a finite length and it converges:

$$\lim_{t\to\infty}\{\theta(t), \omega(t), E(t)\} = \{\theta_\infty, \omega_\infty, E_\infty\}.$$

• Step 3: the velocity vanishes. Using Lemma 5.2 and (5.18) we can find

$$\lim_{t\to\infty}\{\dot\theta(t), \dot\omega(t), \dot E(t)\} = \{0, 0, 0\} \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N.$$

Clearly, $\omega_\infty = 0 \in \mathbb{R}^N$ and $F(\theta_\infty, \omega_\infty, E_\infty) = 0 \in \mathbb{R}^{3N}$. Therefore, we are done with $x_e = \begin{pmatrix} \theta_\infty \\ E_\infty \end{pmatrix}$ and $\nabla V(x_e) = 0$. □

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