MAPPING PROPERTIES OF OPERATOR-VALUED PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. In this paper, we investigate the mapping properties of pseudo-differential operators with operator-valued symbols. We prove the boundedness of regular symbols on Sobolev spaces $H_2^\alpha(\mathbb{R}^d; L_2(\mathcal{M}))$ and Besov spaces $B_{p,q}^\alpha(\mathbb{R}^d; L_p(\mathcal{M}))$ for $\alpha \in \mathbb{R}$ and $1 \le p, q \le \infty$, as well as the boundedness of forbidden symbols on $H_2^\alpha(\mathbb{R}^d; L_2(\mathcal{M}))$ and $B_{p,q}^\alpha(\mathbb{R}^d; L_p(\mathcal{M}))$ for $\alpha > 0$ and $1 \le p, q \le \infty$. Thanks to the smooth atomic decomposition of the operator-valued Triebel-Lizorkin spaces $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ obtained in our previous paper, we establish the $F_1^{\alpha,c}$-regularity of regular symbols for every $\alpha \in \mathbb{R}$, and the $F_1^{\alpha,c}$-regularity of forbidden symbols for $\alpha > 0$. As applications, we obtain the same results on the usual and quantum tori.

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0. INTRODUCTION

Pseudo-differential operators were first explicitly defined by Kohn-Nirenberg [29] and Hörmander [21] to connect singular integrals and differential operators. They can be viewed as generalizations of Fourier multipliers, i.e., those operators acting on functions of variable $s \in \mathbb{R}^d$, formally determined by

$$T(e^{2\pi i s\cdot\xi}) = \sigma(\xi)e^{2\pi i s\cdot\xi}, \quad \forall\, \xi \in \mathbb{R}^d.$$

In this sense, $\sigma(\xi)$ is called the symbol of the operator T. If T is one of those more general operators, it is characterized by the symbol $\sigma(s, \xi)$, which is now a function of s as well as ξ, i.e.,

$$T(e^{2\pi i s\cdot\xi}) = \sigma(s, \xi)e^{2\pi i s\cdot\xi}.$$

Using the inverse Fourier transform, this characterization looks like

$$Tf(s) = \int_{\mathbb{R}^d} \sigma(s, \xi)\widehat{f}(\xi)e^{2\pi i s\cdot\xi}d\xi. \tag{0.1}$$

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To emphasize the role of the symbol σ, we often write T as T_σ. And we call this T_σ a pseudo-differential operator.

Here are some examples of pseudo-differential operators. If σ is independent of the variable s, then we go back to the Fourier multiplier mentioned above. On the other hand, if σ is independent of the variable ξ, then by (0.1), we get $Tf(s) = \sigma(s) \cdot f(s)$, the pointwise multiplication. To give an example of pseudo-differential operator whose symbol is a function of both s and ξ, we consider the partial differential operator $L = \sum_{|m|_1 \leq k} a_m(s) D_s^m$, where $m \in \mathbb{N}_0^d$ and $|m|_1 = m_1 + \cdots + m_d$. This time, by (0.1) again, we know that the symbol of L is

$$\sigma(s,\xi) = \sum_{|m|_1 \leq k} a_m(s)(2\pi \mathrm{i}\xi)^m.$$

For a general symbol σ, T_σ may be thought as a limit of linear combinations of operators composed by pointwise multiplications and Fourier multipliers.

The study of pseudo-differential operators connects the partial differential operators with harmonic analysis. More precisely, the regularity of the solutions of a PDE corresponds to the boundedness of the related pseudo-differential operator on some function spaces. This amounts to one of the most important problems in pseudo-differential operator theory: the mapping properties of these operators on various function spaces. Given $n \in \mathbb{R}$ and $0 \leq \delta, \rho \leq 1$, denote by $S^n_{\rho,\delta}$ the Hörmander class of symbols, consisting of all infinitely differentiable functions $\sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ such that

$$|D_s^\gamma D_\xi^\beta \sigma(s,\xi)| \leq C_{\gamma,\beta}(1+|\xi|)^{n+\delta|\gamma|_1-\rho|\beta|_1} \tag{0.2}$$

for all $s, \xi \in \mathbb{R}^d$. One may ask which kind of symbol classes give pseudo-differential operators that are bounded on L_p-spaces, Sobolev, Besov, Hardy or Triebel-Lizorkin spaces. In general, it is known that pseudo-differential operators are not necessarily bounded on the classical Hardy space $\mathcal{H}_1(\mathbb{R}^d)$, or homogeneous Besov and Triebel-Lizorkin spaces. As a result, when studying the mapping properties of pseudo-differential operators, one usually focuses on inhomogeneous function spaces, such as the local Hardy spaces $\mathrm{h}_p(\mathbb{R}^d)$ defined by Goldberg [17] which provides the inhomogeneous version of [16], or inhomogeneous Besov and Triebel-Lizorkin spaces (see Triebel [52] and [53] for the definitions). For details on these results in the classical setting, we refer to [7, 9, 12, 44, 46, 49, 53].

In the noncommutative setting, this line of research started with Connes and Baaj's work [13, 4] on pseudo-differential calculus for C^*-dynamical systems, which intended to extend the Atiyah-Singer index theorem [3] for Lie group actions on C^*-algebras. At that time, due to the fact that very little had been done about the analytic aspect, the work of Connes and his collaborators did not include L_p-estimates for parametrices or error terms. Recently, inspired by the development of noncommutative harmonic analysis, a lot of progress has been made on Fourier multiplier theory and Calderón-Zygmund theory on noncommutative L_p spaces, thanks to the efforts of many researchers [10, 11, 19, 20, 24, 26, 28, 34, 37, 38, 43, 57, 58, 59]. But so far, the mapping properties of pseudo-differential operators are rarely studied which seems to be a good candidate to connect the noncommutative harmonic analysis with the noncommutative differential geometry [14].

As we know, pseudo-differential operators have substantial impact on linear and non-linear PDEs [47, 48]. In the noncommutative setting, an important motivation for us to investigate the pseudo-differential operators is their potential applications on noncommutative PDEs. Studying their mapping properties on the most fundamental noncommutative manifold quantum tori is surely a good starting point for us. Our strategy is using the transference method [11, 37] to transfer the analysis on fully noncommutative algebras to the case of semicommutative algebras, specifically, from quantum tori to spaces of bounded functions defined on $\mathbb{R}^d$ or $\mathbb{T}^d$ with values in some von Neumann algebras.

In this paper, we consider the boundedness of noncommutative pseudo-differential operators on operator-valued function spaces and then apply the corresponding results to quantum tori. Our definition of symbol classes is modelled on the classical definition by Hörmander; the idea is to consider those operator-valued functions $\sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{M}$ satisfying (0.2) with operator norms in place of absolute values of the derivatives of σ. Here $\mathcal{M}$ is a von Neumann algebra. If $f : \mathbb{R}^d \to L_1(\mathcal{M}) + \mathcal{M}$ is a good enough function, we can consider the action of a pseudo-differential operator with symbol σ on this f. Because of the noncommutativity, we have two different actions:

$$T_\sigma^c f(s)=\int_{\mathbb{R}^d}\sigma(s,\xi)\widehat{f}(\xi)e^{2\pi \mathrm{i} s\cdot\xi}d\xi \tag{0.3}$$

and

$$T_\sigma^r f(s)=\int_{\mathbb{R}^d}\widehat{f}(\xi)\sigma(s,\xi)e^{2\pi \mathrm{i} s\cdot\xi}d\xi.$$

We will mainly work on the column operators T_σ^c and establish their mapping properties on (potential) Sobolev spaces $H_p^\alpha(\mathbb{R}^d;L_p(\mathcal{M}))$ and Besov spaces $B_{p,q}^\alpha(\mathbb{R}^d;L_p(\mathcal{M}))$, as well as local Hardy spaces $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ and inhomogeneous Triebel-Lizorkin spaces $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$. The Sobolev and Besov spaces are defined in a similar way as their Banach-valued counterparts (see section 3 for concrete definitions in the operator-valued setting), while the local Hardy spaces and Triebel-Lizorkin spaces are introduced and studied in our recent papers [55] and [56].

Now we state the main results of this paper, and briefly describe the ingredients of the proofs. We concentrate on the pseudo-differential operators with operator-valued symbols in $S_{1,\delta}^n$, the class of infinitely differentiable functions $\sigma:\mathbb{R}^d\times\mathbb{R}^d\to\mathcal{M}$ such that

$$\|D_s^\gamma D_\xi^\beta\sigma(s,\xi)\|_{\mathcal{M}}\leq C_{\gamma,\beta}(1+|\xi|)^{n+\delta|\gamma|_1-|\beta|_1},\quad \forall\, s,\xi\in\mathbb{R}^d.$$

The first part of the results is about the Sobolev and Besov spaces. Let $\sigma\in S_{1,\delta}^0$.

i) If $0\leq\delta<1$, T_σ^c is bounded on the Sobolev space $H_2^\alpha(\mathbb{R}^d;L_2(\mathcal{M}))$ for any $\alpha\in\mathbb{R}$.
ii) If $\delta=1$, T_σ^c is bounded on $H_2^\alpha(\mathbb{R}^d;L_2(\mathcal{M}))$ for any $\alpha>0$.
iii) If $0\leq\delta<1$, T_σ^c is bounded on the Besov space $B_{p,q}^\alpha(\mathbb{R}^d;L_p(\mathcal{M}))$ for any $1\leq p,q\leq\infty$ and $\alpha\in\mathbb{R}$.
iv) If $\delta=1$, T_σ^c is bounded on $B_{p,q}^\alpha(\mathbb{R}^d;L_p(\mathcal{M}))$ for any $1\leq p,q\leq\infty$ and $\alpha>0$.

The proof of the L_2 regularity in i) is similar to the corresponding classical result, relying heavily on the Cotlar-Stein Orthogonality Lemma. But if $\sigma\in S_{1,1}^0$, we no longer have this L_2 regularity; we prove the boundedness of T_σ^c on the Sobolev spaces $H_2^\alpha(\mathbb{R}^d;L_2(\mathcal{M}))$ for $\alpha>0$ instead. To demonstrate this boundedness on Sobolev spaces, we will have to treat the kernels of the dyadic pieces of symbols, and use the relation between Sobolev and Besov spaces. The same trick will lead to the boundedness of T_σ^c on Besov spaces.

The regularities of pseudo-differential operators on local Hardy spaces and Triebel-Lizorkin spaces are much more complicated. The main part of the proof concerns the case $p=1$ for both kinds of spaces. Compared to the standard proof of the boundedness on Hardy spaces of a usual Calderón-Zygmund operator with a commutative or noncommutative convolution kernel, the present proof is much subtler and more technical. We need a careful analysis of the pseudo-differential operator acting on smooth (sub)atoms given in [56]. Our results are the following.

i) If $0\leq\delta<1$ and $\sigma\in S_{1,\delta}^0$, T_σ^c is bounded on $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ for any $1\leq p\leq\infty$ and $\alpha\in\mathbb{R}$.
ii) If $\sigma\in S_{1,1}^0$, T_σ^c is bounded on $F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ for every $\alpha>0$.

For $0\leq\delta<1$, we first prove $T_\sigma: F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})\to F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ is bounded. Since the adjoint of $\sigma\in S_{1,\delta}^0$ still belongs to $S_{1,\delta}^0$ when $\delta<1$, we will deduce the boundedness on $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ from duality and interpolation. And for $\sigma\in S_{1,\delta}^n$, we can use the Bessel potential of order n to connect T_σ^c with $T_{\sigma'}^c$ for $\sigma'\in S_{1,\sigma}^0$, i.e. $T_\sigma^c=T_{\sigma'}^c\circ J^n$, so as to get the boundedness of T_σ^c from $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ to $F_p^{\alpha-n,c}(\mathbb{R}^d,\mathcal{M})$. For the case $\delta=1$, in order to to get the boundedness on $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ for $p>1$, we need more assumptions on the symbol.

We then apply the outcome to the usual and quantum tori, and obtain parallel results in both cases.

Our regularity results on Sobolev and Besov spaces can be viewed as a particular case of more general Banach-valued inequalities for pseudo-differential operators. Based on the work of Weis [54], Portal and Štrkalj [42] proved that the pseudo-differential operators with operator-valued symbols $\sigma(s,\xi):\mathbb{R}^d\times\mathbb{R}^d\to B(X)$ are bounded on the Bochner integrable spaces $L_p(\mathbb{R}^d;X)$, under the essential assumptions (1) σ is R-bounded (defined in [54]), (2) the Banach space X is a UMD space (see [8]). Since the noncommutative L_p spaces are UMD spaces when $1<p<\infty$, Portal and Štrkalj's results apply to $L_p(L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M})=L_p(\mathbb{R}^d;L_p(\mathcal{M}))$ immediately. However, because of the R-boundedness assumption, their results do not cover the symbol classes in this paper.

The situation for Banach-valued Besov spaces is more satisfactory, where neither R-boundedness nor UMD condition are needed, see [1] for results on Fourier multipliers. More recently, in [5] the authors proved the mapping properties for operator-valued pseudo-differential operators on toroidal Besov spaces $B^{\alpha}_{p,q}(\mathbb{T}^d;X)$ for $1\le p,q\le\infty$ and an arbitrary Banach space X, which covers the corresponding results in section 7. The regularity results on Sobolev and Besov spaces obtained in our paper should be known to experts, but have not been studied systematically in the literature. For the sake of completeness and for our use in the proofs of the regularity results on Triebel-Lizorkin spaces, we include those results, and give proper proofs of them.

Contrary to the Sobolev or Besov case, our regularity results on Triebel-Lizorkin spaces do not follow from any pseudo-differential operator theory in the Banach-valued setting. Let us illustrate this at the level of Hardy spaces. For an $\mathcal{M}$-valued function f on $\mathbb{R}^d$, given a Littlewood-Paley decomposition $(\varphi_j)_{j\ge0}$ on $\mathbb{R}^d$, the Hardy space norm used in our paper is given by

$$\|f\|_{\mathrm{h}_p}=\begin{cases}\inf\limits_{f=g+h}\Big\{\big\|(\sum_{j\ge0}|\varphi_j*g|^2)^{\frac12}\big\|_p+\big\|(\sum_{j\ge0}|\varphi_j*h^*|^2)^{\frac12}\big\|_p\Big\} & \text{if}\quad 1\le p\le2\\ \max\Big\{\big\|(\sum_{j\ge0}|\varphi_j*f|^2)^{\frac12}\big\|_p,\ \big\|(\sum_{j\ge0}|\varphi_j*f^*|^2)^{\frac12}\big\|_p\Big\} & \text{if}\quad 2<p<\infty.\end{cases}$$

By the noncommutative Khintchine inequalities [31, 32], this norm is equivalent to $\|\sum_{j\ge0}r_j\cdot\varphi_j*f\|_{L_p(\Omega\times\mathbb{R}^d;L_p(\mathcal{M}))}$, where $\{r_j\}_{j\ge0}$ is a Rademacher sequence on some probability space (Ω,P). It seems that this later norm has not been studied so far in the literature.

Let us mention that independently and at the same time, González-Pérez, Junge and Parcet developed in [18] the pseudo-differential theory in quantum Euclidean spaces that are the non compact analogues of quantum tori. Although the two papers overlap in some ways, they are very different in nature in regard to both arguments and results. The arguments of [18] are based on a careful analysis of the L_2 and semigroup BMO cases (the latter is defined in [27]), while our proof in the case $p=1$ (the main case) relies entirely on the atomic decomposition of $F^{\alpha,c}_1(\mathbb{R}^d,\mathcal{M})$ obtained in [56]. Interestingly, as far as the results are concerned, our results deal with the asymmetric situation of boundedness of T^c_σ (T^r_σ) on column (row) Triebel-Lizorkin spaces $F^{\alpha,c}_p(\mathbb{R}^d,\mathcal{M})$ ($F^{\alpha,r}_p(\mathbb{R}^d,\mathcal{M})$) with $1\le p\le\infty$. However, our methods do not yield the $F^{\alpha,r}_p$-regularity of T^c_σ nor the $F^{\alpha,c}_p$-regularity of T^r_σ, thus in particular we are not able to get the L_p-regularity of T^c_σ since $F^{\alpha}_p(\mathbb{R}^d,\mathcal{M})$ coincides with $L_p(L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M})$ when $\alpha=0$ and $1<p<\infty$. In [18], in the quantum Euclidean setting, the authors define the class of symbols $\Sigma^0_{\rho,\delta}$ which eliminates this asymmetry, and obtain the boundedness of the corresponding pseudo-differential operators on L_p-spaces with $1<p<\infty$. Coming back to the commutative case, $\Sigma^0_{\rho,\delta}$ reduces to the case of classical Hörmander symbols, while in the fully noncommutative setting, $\Sigma^0_{\rho,\delta}$ is strictly smaller than $S^0_{\rho,\delta}$. Thus the L_p-regularity of T^c_σ (or T^r_σ) is still unsolved for the whole class $S^0_{\rho,\delta}$.

The paper is organized as follows. In section 1, we introduce some elementary notation and knowledge on noncommutative L_p-spaces, and the definitions of local Hardy spaces in [55] and inhomogeneous Triebel-Lizorkin spaces in [56]. Then we present the smooth atomic decompositions of these spaces obtained in [56]. In section 2, we give the concrete definitions and some easily deduced useful facts on operator-valued pseudo-differential operators. In section 3 we prove the mapping properties of pseudo-differential operators on Sobolev and Besov spaces. Section 4 is devoted to the study of the local mapping properties of pseudo-differential operators, i.e. their action on atoms. In sections 5 and 6, we prove the mapping properties of pseudo-differential operators with regular and forbidden symbols respectively. The last section presents applications to the usual and quantum tori.

We close this introduction section by the following convention. Throughout, we will use the notation $A\lesssim B$, which is an inequality up to a constant: $A\le cB$ for some constant $c>0$. The relevant constants in all such inequalities may depend on the dimension d, the test functions φ or Φ, or p, etc., but never on the function f in consideration. The equivalence $A\approx B$ will mean $A\lesssim B$ and $B\lesssim A$ simultaneously.

1. Preliminaries on noncommutative analysis

We begin with an introduction of notation and basic knowledge on vector-valued Fourier analysis, i.e., Fourier analysis on functions with values in a Banach spaces X. Let $\mathcal{S}(\mathbb{R}^d;X)$ be the space of X-valued rapidly decreasing and infinitely differentiable functions on $\mathbb{R}^d$ with the standard Fréchet topology. In particular, $\mathcal{S}(\mathbb{R}^d;\mathbb{C})$ is simply denoted as $\mathcal{S}(\mathbb{R}^d)$. Let $\mathcal{S}'(\mathbb{R}^d;X)$ be the space of continuous linear maps from $\mathcal{S}(\mathbb{R}^d)$ to X; the elements of $\mathcal{S}'(\mathbb{R}^d;X)$ are the so-called X-valued tempered distributions. All operations on $\mathcal{S}(\mathbb{R}^d)$ such as derivation, convolution and Fourier transform transfer to $\mathcal{S}'(\mathbb{R}^d;X)$ in the usual way. On the other hand, $L_p(\mathbb{R}^d;X)$ naturally embeds into $\mathcal{S}'(\mathbb{R}^d;X)$ for $1\le p\le\infty$, where $L_p(\mathbb{R}^d;X)$ stands for the space of strongly p-integrable functions from $\mathbb{R}^d$ to X. By this definition, Fourier transform and Fourier multipliers on $\mathbb{R}^d$ extend to vector-valued tempered distributions in a natural way.

We give some typical Fourier multipliers that will be frequently used in the sequel. For a real number α, the Bessel potential is the operator $J^\alpha=(1-(2\pi)^{-2}\Delta)^{\frac{\alpha}{2}}$ defined on $\mathcal{S}'(\mathbb{R}^d;X)$, where Δ denotes the Laplacian on $\mathbb{R}^d$. If $\alpha=1$, we will abbreviate J^1 as J. We denote also $J_\alpha(\xi)=(1+|\xi|^2)^{\frac{\alpha}{2}}$ on $\mathbb{R}^d$. It is the symbol of the Fourier multiplier J^α. Recall also the symbols of Littlewood-Paley decomposition on $\mathbb{R}^d$, which are used to define Besov and Triebel-Lizorkin spaces. Fix a Schwartz function φ on $\mathbb{R}^d$ satisfying:

$$\begin{cases}\operatorname{supp}\varphi\subset\{\xi:\frac12\le|\xi|\le2\}.\\ \varphi>0 \text{ on } \{\xi:\frac12<|\xi|<2\},\\ \sum_{k\in\mathbb{Z}}\varphi(2^{-k}\xi)=1,\forall\,\xi\neq0.\end{cases}\tag{1.1}$$

Given $k\in\mathbb{N}$, let φ_k be the function whose Fourier transform is equal to $\varphi(2^{-k}\cdot)$ and φ_0 be the function whose Fourier transform is equal to $1-\sum_{k>0}\varphi(2^{-k}\cdot)$. Then $\{\varphi_k\}_{k\ge0}$ gives a Littlewood-Paley decomposition on $\mathbb{R}^d$ such that

$$\operatorname{supp}\widehat{\varphi}_k\subset\{\xi\in\mathbb{R}^d:2^{k-1}\le|\xi|\le2^{k+1}\},\ \forall\,k\in\mathbb{N},\quad \operatorname{supp}\widehat{\varphi}_0\subset\{\xi\in\mathbb{R}^d:|\xi|\le2\}\tag{1.2}$$

and that

$$\sum_{k=0}^{\infty}\widehat{\varphi}_k(\xi)=1,\quad\forall\,\xi\in\mathbb{R}^d.\tag{1.3}$$

Other than the above Littlewood-Paley decomposition, we will need another kind of resolution of the unit on $\mathbb{R}^d$ (see [46, Section VII.2.4]). Let $\mathcal{X}_0$ be a nonnegative infinitely differentiable function on $\mathbb{R}^d$ such that $\operatorname{supp}\mathcal{X}_0\subset 2Q_{0,0}$, and $\sum_{k\in\mathbb{Z}^d}\mathcal{X}_0(s-k)=1$ for every $s\in\mathbb{R}^d$. Here $Q_{0,0}$ is the unit cube centered at the origin, and $2Q_{0,0}$ is the cube with the same center, but twice the side length; see the end of this section for notation of general cubes. Set $\mathcal{X}_k=\mathcal{X}_0(\cdot-k)$. Then each $\mathcal{X}_k$ is supported in the cube $2Q_{0,k}=k+2Q_{0,0}$, and all $\mathcal{X}_k$'s form a smooth resolution of the unit:

$$1=\sum_{k\in\mathbb{Z}^d}\mathcal{X}_k(s),\quad\forall\,s\in\mathbb{R}^d.\tag{1.4}$$

This smooth resolution of the unit will often be used to divide functions or distributions into small pieces, which have the same smoothness as before, but have compact supports additionally.

1.1. **Noncommutative L_p-spaces.** Let us turn to the setting of operator-valued analysis, where the above involved Banach spaces X are required to have some operator space structure now. In this paper, all function spaces in consideration are based on the noncommutative L_p-spaces associated to $(\mathcal{M},\tau)$, where $\mathcal{M}$ is a von Neumann algebra τ is a normal semifinite faithful trace, and $1\le p\le\infty$. The norm of $L_p(\mathcal{M})$ will be often denoted simply by $\|\cdot\|_p$. But if different L_p-spaces appear in a same context, we will sometimes precise the respective L_p-norms in order to avoid possible ambiguity. The reader is referred to [41, 60, 25] for more information on noncommutative L_p-spaces. These noncommutative L_p-spaces are equipped with their natural operator space structure introduced by Pisier [39, 40]. The structure on $L_1(\mathcal{M})$ is defined as the one induced by the opposite of dual space $(\mathcal{M}')^{\mathrm{op}}$. For $1<p<\infty$, the natural operator space structure on $L_p(\mathcal{M})$ is given by the family of norms determined by the complex interpolation $M_n(L_p(\mathcal{M}))=\big(M_n(\mathcal{M}),M_n(L_1(\mathcal{M}))\big)_{\frac1p}$, where the norm of $M_n(\mathcal{M})\subset M_n(B(H))$ is induced by the one on $M_n(B(H))\cong B(\ell_2^n(H))$.

We will also need Hilbert space-valued noncommutative L_p-spaces (see [23] for more details). Let H be a Hilbert space and $v\in H$ with $\|v\|=1$. Let p_v be the orthogonal projection onto the one-dimensional subspace generated by v. Define

$$L_p(\mathcal{M};H^r)=(p_v\otimes 1_{\mathcal{M}})L_p(B(H)\overline{\otimes}\mathcal{M}) \text{ and } L_p(\mathcal{M};H^c)=L_p(B(H)\overline{\otimes}\mathcal{M})(p_v\otimes 1_{\mathcal{M}}).$$

These are the row and column noncommutative L_p-spaces, which are 1-complemented subspaces of $L_p(B(H)\overline{\otimes}\mathcal{M})$.

In most part of this paper, we are interested in operator-valued functions. The involved von Neumann algebra is the semi-commutative algebra $L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M}$ with tensor trace, denoted by $\mathcal{N}$ in the sequel. We will frequently use the following Cauchy-Schwarz type inequality,

$$\Big|\int_{\mathbb{R}^d}\phi(s)f(s)ds\Big|^2\le\int_{\mathbb{R}^d}|\phi(s)|^2ds\int_{\mathbb{R}^d}|f(s)|^2ds, \tag{1.5}$$

where $\phi:\mathbb{R}^d\to\mathbb{C}$ and $f:\mathbb{R}^d\to L_1(\mathcal{M})+\mathcal{M}$ are functions such that all integrations of the above inequality make sense. Here for an operator x, $|x|^2$ denotes x^*x. (1.5) is an easy consequence of the convexity of the operator-valued function: $x\mapsto|x|^2$, and "$\le$" is understood as the partial order in the positive cone of $\mathcal{M}$. We will also require the operator-valued version of the Plancherel formula. For sufficiently nice functions $f:\mathbb{R}^d\to L_1(\mathcal{M})+\mathcal{M}$, for example, for $f\in L_2(\mathbb{R}^d)\otimes L_2(\mathcal{M})$, we have

$$\int_{\mathbb{R}^d}|f(s)|^2ds=\int_{\mathbb{R}^d}|\widehat{f}(\xi)|^2d\xi. \tag{1.6}$$

1.2. **Inhomogeneous Triebel-Lizorkin spaces.** We follow the presentation in [56], to give the definition of inhomogeneous Triebel-Lizorkin spaces. Let $1\le p<\infty$ and $\alpha\in\mathbb{R}$, and φ be the Schwartz function determined by (1.1). The column Triebel-Lizorkin space $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ is defined by

$$F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})=\{f\in\mathcal{S}'(\mathbb{R}^d;L_1(\mathcal{M})+\mathcal{M}):\|f\|_{F_p^{\alpha,c}}<\infty\},$$

where

$$\|f\|_{F_p^{\alpha,c}}=\Big\|\Big(\sum_{j\ge0}2^{2j\alpha}|\varphi_j*f|^2\Big)^{\frac12}\Big\|_{L_p(\mathcal{N})}.$$

The row space $F_p^{\alpha,r}(\mathbb{R}^d,\mathcal{M})$ consists of all f such that $f^*\in F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$, equipped with the norm $\|f\|_{F_p^{\alpha,r}}=\|f^*\|_{F_p^{\alpha,c}}$. The mixture space $F_p^{\alpha}(\mathbb{R}^d,\mathcal{M})$ is defined to be

$$F_p^{\alpha}(\mathbb{R}^d,\mathcal{M})=\begin{cases}F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})+F_p^{\alpha,r}(\mathbb{R}^d,\mathcal{M}) & \text{if} \quad 1\le p\le 2\\ F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})\cap F_p^{\alpha,r}(\mathbb{R}^d,\mathcal{M}) & \text{if} \quad 2<p<\infty,\end{cases}$$

equipped with

$$\|f\|_{F_p^{\alpha}}=\begin{cases}\inf\limits_{f=g+h}\{\|g\|_{F_p^{\alpha,c}}+\|h\|_{F_p^{\alpha,r}}\} & \text{if} \quad 1\le p\le 2\\ \max\{\|f\|_{F_p^{\alpha,c}},\|f\|_{F_p^{\alpha,r}}\} & \text{if} \quad 2<p<\infty.\end{cases}$$

If $p=\infty$, define $F_\infty^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ as the space of all $f\in\mathcal{S}'(\mathbb{R}^d;\mathcal{M})$ such that

$$\|f\|_{F_\infty^{\alpha,c}}=\|\varphi_0*f\|_{\mathcal{N}}+\sup_{|Q|<1}\Big\|\frac{1}{|Q|}\int_Q\sum_{j\ge-\log_2(l(Q))}2^{2j\alpha}|\varphi_j*f(s)|^2ds\Big\|_{\mathcal{M}}^{\frac12}<\infty,$$

where Q denotes any cube in $\mathbb{R}^d$, $|Q|$ its volume, and $l(Q)$ its side length. Let $1\le p<\infty$, $\alpha\in\mathbb{R}$ and q be the conjugate index of p. Then the dual space of $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ coincides isomorphically with $F_q^{-\alpha,c}(\mathbb{R}^d,\mathcal{M})$.

The Triebel-Lizorkin spaces form an interpolation scale with respect to the complex interpolation method [6]: For $\alpha_0,\alpha_1\in\mathbb{R}$ and $1<p<\infty$, we have

$$\big(F_\infty^{\alpha_0,c}(\mathbb{R}^d,\mathcal{M}),F_1^{\alpha_1,c}(\mathbb{R}^d,\mathcal{M})\big)_{\frac1p}=F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M}),\quad \alpha=(1-\frac1p)\alpha_0+\frac{\alpha_1}{p}.$$

See [56] for the proof of this interpolation.

When $\alpha=0$ and $1\le p<\infty$, it is proved in [56] that $F_p^{0,c}(\mathbb{R}^d,\mathcal{M})=\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ with equivalent norms, where $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ (see [55] for the definition) are the local analogues of the operator-valued Hardy spaces defined in [33]. The lifting property of Triebel-Lizorkin spaces states that, for any $\beta\in\mathbb{R}$, J^β is an isomorphism between $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ and $F_p^{\alpha-\beta,c}(\mathbb{R}^d,\mathcal{M})$. In particular, J^α is an isomorphism between $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$. In this sense, these Triebel-Lizorkin spaces can be viewed as an extension of local Hardy spaces. Moreover, when $1 < p < \infty$, we have, with equivalent norms,

$$L_p(\mathcal{N}) = \mathrm{h}_p(\mathbb{R}^d, \mathcal{M}) = F_p^0(\mathbb{R}^d, \mathcal{M}), \tag{1.7}$$

where $\mathcal{N} = L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M}$. In the classical Euclidean setting, when $1 < p < \infty$, the local Hardy space $\mathrm{h}_p(\mathbb{R}^d)$ is equivalent to the usual Hardy space $\mathcal{H}_p(\mathbb{R}^d)$ as well as $L_p(\mathbb{R}^d)$, while when $p = 1$ one has the strict inclusions $\mathcal{H}_1(\mathbb{R}^d) \subset \mathrm{h}_1(\mathbb{R}^d) \subset L_1(\mathbb{R}^d)$; see [17] for more details.

1.3. **Atomic decompositions.** We begin with the case $\alpha = 0$, i.e., the atomic decomposition of local Hardy space $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$. Much as in the classical case, the atomic decomposition of $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$ can be deduced from the h_1-bmo duality. The following definition of atoms is given in [55].

Definition 1.1. *Let Q be a cube in $\mathbb{R}^d$ with $|Q| \le 1$. If $|Q| = 1$, an h_1^c-atom associated with Q is a function $a \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$ such that*

- $\operatorname{supp} a \subset Q$;
- $\tau\big(\int_Q |a(s)|^2 ds\big)^{\frac12} \le |Q|^{-\frac12}$.

If $|Q| < 1$, we assume additionally:

- $\int_Q a(s) ds = 0$.

Let $\mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d, \mathcal{M})$ be the space of all f admitting a representation of the form

$$f = \sum_{j=1}^\infty \lambda_j a_j,$$

where the a_j's are h_1^c-atoms and $\lambda_j \in \mathbb{C}$ such that $\sum_{j=1}^\infty |\lambda_j| < \infty$. The above series converges in the sense of distribution. We equip $\mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d, \mathcal{M})$ with the following norm:

$$\|f\|_{\mathrm{h}_{1,\mathrm{at}}^c} = \inf\{\sum_{j=1}^\infty |\lambda_j| : f = \sum_{j=1}^\infty \lambda_j a_j;\ a_j\text{'s are } \mathrm{h}_1^c \text{ -atoms}, \ \lambda_j \in \mathbb{C}\}.$$

It is proved in [55] that

$$\mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d, \mathcal{M}) = \mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M}) \tag{1.8}$$

with equivalent norms. It is also evident in the proof of (1.8) given in [55] that, in Definition 1.1, we can replace the support Q of atoms by any bounded multiple of Q.

Before proceeding further, we point out that throughout the paper, we will use the following notations for cubes in $\mathbb{R}^d$: For any cube $Q \subset \mathbb{R}^d$ and any positive integer λ, λQ is the cube with the same center as Q but side length scaled by a factor λ; for $s \in \mathbb{R}^d$, $s + Q$ denotes the cube obtained by shifting Q by the vector $s = (s_1, \cdots, s_d)$.

Let us introduce the smooth atomic decomposition of $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$, which will be a key ingredient to obtain the boundedness of pseudo-differential operators on $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$. This decomposition is an extension as well as an improvement of the atomic decomposition of $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$ in (1.8). Compared to (1.8), the smoothness of atoms is improved and subatoms enter in the game.

For every $l = (l_1, \cdots, l_d) \in \mathbb{Z}^d$, $\mu \in \mathbb{N}_0$, we define $Q_{\mu,l}$ in $\mathbb{R}^d$ to be the cubes centered at $2^{-\mu} l$, and with side length $2^{-\mu}$. For instance, $Q_{0,0} = [-\frac12, \frac12)^d$ is the unit cube centered at the origin. Let $\mathbb{D}_d$ be the collection of all the cubes $Q_{\mu,l}$ defined above. We write $(\mu, l) \le (\mu', l')$ if

$$\mu \ge \mu' \quad \text{and} \quad Q_{\mu,l} \subset 2Q_{\mu',l'}.$$

For $a \in \mathbb{R}$, let $a_+ = \max\{a, 0\}$ and $[a]$ the largest integer less than or equal to a. Denote $|\gamma|_1 = \gamma_1 + \cdots + \gamma_d$ and $D^\gamma = \partial_1^{\gamma_1} \cdots \partial_d^{\gamma_d}$ for $\gamma \in \mathbb{N}_0^d$, and $s^\beta = s_1^{\beta_1} \cdots s_d^{\beta_d}$ for $s \in \mathbb{R}^d$, $\beta \in \mathbb{N}_0^d$. Recall that J^α is the Bessel potential of order α.

Definition 1.2. *Let $\alpha \in \mathbb{R}$, and let K and L be two integers such that*

$$K \ge ([\alpha] + 1)_+ \quad \textit{and} \quad L \ge \max\{[-\alpha], -1\}.$$

i) *A function $b \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$ is called an $(\alpha, 1)$-atom if*
   - $\operatorname{supp} b \subset 2Q_{0,k}$, $k \in \mathbb{Z}^d$;
   - $\tau(\int_{\mathbb{R}^d} |D^\gamma b(s)|^2 ds)^{\frac12} \le 1, \ \forall \gamma \in \mathbb{N}_0^d, \ |\gamma|_1 \le K$.

ii) *Let* $Q = Q_{\mu,l} \in \mathbb{D}_d$, *a function* $a \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$ *is called an* (α, Q)*-subatom if*
- $\operatorname{supp} a \subset 2Q$;
- $\tau(\int_{\mathbb{R}^d} |D^\gamma a(s)|^2 ds)^{\frac{1}{2}} \leq |Q|^{\frac{\alpha}{d} - \frac{|\gamma|_1}{d}}, \ \forall \gamma \in \mathbb{N}_0^d, \ |\gamma|_1 \leq K$;
- $\int_{\mathbb{R}^d} s^\beta a(s) ds = 0, \ \forall \beta \in \mathbb{N}_0^d, \ |\beta|_1 \leq L$.

iii) *A function* $g \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$ *is called an* $(\alpha, Q_{k,m})$*-atom if*

$$\tau\Big(\int_{\mathbb{R}^d} |J^\alpha g(s)|^2 ds\Big)^{\frac{1}{2}} \leq |Q_{k,m}|^{-\frac{1}{2}} \quad \text{and} \quad g = \sum_{(\mu,l) \leq (k,m)} d_{\mu,l} a_{\mu,l}, \tag{1.9}$$

for some $k \in \mathbb{N}_0$ *and* $m \in \mathbb{Z}^d$, *where the* $a_{\mu,l}$*'s are* $(\alpha, Q_{\mu,l})$*-subatoms and the* $d_{\mu,l}$*'s are complex numbers such that*

$$\Big(\sum_{(\mu,l) \leq (k,m)} |d_{\mu,l}|^2\Big)^{\frac{1}{2}} \leq |Q_{k,m}|^{-\frac{1}{2}}.$$

We have obtained in [56, Theorem 5.7] the following smooth atomic decomposition:

Theorem 1.3. *Let* $\alpha \in \mathbb{R}$ *and* K, L *be two integers fixed as in Definition 1.2. Then any* $f \in F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ *can be represented as*

$$f = \sum_{j=1}^{\infty} \big(\mu_j b_j + \lambda_j g_j\big), \tag{1.10}$$

where the b_j*'s are* $(\alpha, 1)$*-atoms, the* g_j*'s are* (α, Q)*-atoms, and* μ_j, λ_j *are complex numbers with*

$$\sum_{j=1}^{\infty} (|\mu_j| + |\lambda_j|) < \infty. \tag{1.11}$$

Moreover, the infimum of (1.11) *with respect to all admissible representations yields an equivalent norm in* $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$.

It is worthwhile to point out that the above K and L can be arbitrarily large, depending on the resolution of the unit used in the proof of Theorem 1.3 given in [56]. In other words, the orders of the smoothness and moment cancellation of the atoms are at our disposal, so that we can require good enough conditions on the atoms. This will be a very important technique in the proofs of our main results.

2. Pseudo-differential operators: definitions and basic properties

We introduce the definitions and some basic properties of pseudo-differential operators in this section. The symbols of pseudo-differential operators considered here are $B(X)$-valued, where X is a Banach space and $B(X)$ denotes the space of all bounded linear operators on X. However, in the later sections, we will only consider those symbols with values in $\mathcal{M}$.

The content of this section is a straightforward generalization to the vector-valued case of the classical theory of pseudo-differential operators, see for instance [46, 47, 48, 45]. Such a generalization already appears in some recent papers, see [42, 5]. So we claim no originality here. Nonetheless, for the sake of completeness, we prefer to include the definitions specifically, and provide complete proofs of the basic properties that will be used in the next sections.

Let $n \in \mathbb{R}$ and $0 \leq \delta, \rho \leq 1$. Then $S^n_{\rho,\delta}$ denotes the collection of all infinitely differentiable functions σ defined on $\mathbb{R}^d \times \mathbb{R}^d$ and with values in $B(X)$, such that for each pair of multi-indices of nonnegative integers γ, β, the inequality

$$\|D_s^\gamma D_\xi^\beta \sigma(s, \xi)\|_{B(X)} \leq C_{\gamma,\beta}(1 + |\xi|)^{n + \delta|\gamma|_1 - \rho|\beta|_1}$$

holds for some constant $C_{\gamma,\beta}$ depending on γ, β and σ. Here again $\gamma = (\gamma_1, \cdots, \gamma_d) \in \mathbb{N}_0^d$, $|\gamma|_1 = \gamma_1 + \cdots + \gamma_d$ and $D_s^\gamma = \frac{\partial^{\gamma_1}}{\partial s_1^{\gamma_1}} \cdots \frac{\partial^{\gamma_d}}{\partial s_d^{\gamma_d}}$.

Definition 2.1. *Let* $\sigma \in S^n_{\rho,\delta}$. *For function* $f \in \mathcal{S}(\mathbb{R}^d; X)$, *the pseudo-differential operator* T_σ *is a mapping* $f \mapsto T_\sigma f$ *given by*

$$T_\sigma f(s) = \int_{\mathbb{R}^d} \sigma(s, \xi) \widehat{f}(\xi) e^{2\pi \mathrm{i} s \cdot \xi} d\xi. \tag{2.1}$$

We call σ *the symbol of* T_σ.

Proposition 2.2. *Let $0 \le \delta, \rho \le 1$ and $n \in \mathbb{R}$. For any $\sigma \in S^n_{\rho,\delta}$, T_σ is continuous on $\mathcal{S}(\mathbb{R}^d; X)$.*

Proof. By integration by parts, for any $s \in \mathbb{R}^d$ and $\gamma \in \mathbb{N}_0^d$, we have

$$\begin{aligned}\|(2\pi \mathrm{i} s)^\gamma T_\sigma f\|_X &= \Big\|(2\pi \mathrm{i} s)^\gamma \int_{\mathbb{R}^d} \sigma(s,\xi)\widehat{f}(\xi) e^{2\pi \mathrm{i} s\cdot\xi} d\xi\Big\|_X \\ &= \Big\| \int_{\mathbb{R}^d} \sigma(s,\xi)\widehat{f}(\xi) D_\xi^\gamma (e^{2\pi \mathrm{i} s\cdot\xi}) d\xi\Big\|_X \\ &= \Big\| \int_{\mathbb{R}^d} D_\xi^\gamma[\sigma(s,\xi)\widehat{f}(\xi)] e^{2\pi \mathrm{i} s\cdot\xi} d\xi\Big\|_X < \infty.\end{aligned}$$

Thus, $T_\sigma f$ is rapidly decreasing. A similar argument works for the partial derivatives of $T_\sigma f$, then we easily check that $T_\sigma f$ maps $\mathcal{S}(\mathbb{R}^d; X)$ continuously to itself. □

Another way to write (2.1) is as a double integral:

$$T_\sigma f(s) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma(s,\xi) f(t) e^{2\pi \mathrm{i}(s-t)\cdot\xi} dt d\xi. \tag{2.2}$$

However, the above ξ-integral does not necessarily converge absolutely, even for $f \in \mathcal{S}(\mathbb{R}^d; X)$. To overcome this difficulty, we will approximate σ by symbols with compact support. To this end, let us fix a compactly supported infinitely differentiable function η defined on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\eta = 1$ near the origin. Set

$$\sigma_j(s,\xi) = \sigma(s,\xi)\eta(2^{-j}s, 2^{-j}\xi), \quad j \in \mathbb{N}. \tag{2.3}$$

Note that σ_j converges pointwise to σ and $\sigma_j \in S^n_{\rho,\delta}$ uniformly in j. Thus, for any $f \in \mathcal{S}(\mathbb{R}^d; X)$, $T_{\sigma_j} f$ converges to $T_\sigma f$ in $\mathcal{S}(\mathbb{R}^d; X)$ as $j \to \infty$. Since the σ_j's have compact supports, formula (2.2) works for $T_{\sigma_j} f(s)$. Then we can define the integral (2.2) as follows:

$$T_\sigma f(s) = \lim_{j\to\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_j(s,\xi) f(t) e^{2\pi \mathrm{i}(s-t)\cdot\xi} dt d\xi. \tag{2.4}$$

Proposition 2.3. *Let $0 \le \delta < 1, 0 \le \rho \le 1$ and $n \in \mathbb{R}$. For any $\sigma \in S^n_{\rho,\delta}$, the adjoint of T_σ is continuous on $\mathcal{S}(\mathbb{R}^d; X^*)$.*

Proof. For any $f \in \mathcal{S}(\mathbb{R}^d; X)$ and $g \in \mathcal{S}(\mathbb{R}^d; X^*)$, by the duality relation

$$\langle T_\sigma f, g\rangle = \langle f, (T_\sigma)^* g\rangle,$$

we check that

$$(T_\sigma)^* g(s) = \lim_{j\to\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_j^*(t,\xi) g(t) e^{2\pi \mathrm{i}(s-t)\cdot\xi} dt d\xi. \tag{2.5}$$

By integration by parts, it is clear that $(T_\sigma)^*$ is continuous on $\mathcal{S}(\mathbb{R}^d; X^*)$. □

Since $\mathcal{S}'(\mathbb{R}^d; X^{**}) = (\mathcal{S}(\mathbb{R}^d; X^*))^*$ (see [51, Section 51] for more details on this duality), in the usual way, we extend T_σ to an operator on $\mathcal{S}'(\mathbb{R}^d; X^{**})$.

Definition 2.4. *Let $f \in \mathcal{S}'(\mathbb{R}^d; X^{**})$. We define $T_\sigma f$ by*

$$\langle T_\sigma f, g\rangle = \langle f, (T_\sigma)^* g\rangle, \quad \forall\, g \in \mathcal{S}(\mathbb{R}^d; X^*).$$

By Proposition 2.3, $(T_\sigma)^* g \in \mathcal{S}(\mathbb{R}^d; X^*)$ whenever $g \in \mathcal{S}(\mathbb{R}^d; X^*)$. So the bracket on the right hand side of the above definition is well defined. Therefore, $T_\sigma f$ is well defined, and takes value in $\mathcal{S}'(\mathbb{R}^d; X^{**})$ as well.

Proposition 2.5. *Let $0 \le \delta < 1, 0 \le \rho \le 1$ and $n \in \mathbb{R}$. For any $\sigma \in S^n_{\rho,\delta}$, T_σ is continuous on $\mathcal{S}'(\mathbb{R}^d; X^{**})$.*

Proof. For any $f \in \mathcal{S}'(\mathbb{R}^d; X^{**})$, we take a sequence (f_j) such that $f_j \to f$ in $\mathcal{S}'(\mathbb{R}^d; X^{**})$. Then we have

$$\langle T_\sigma f_j, g\rangle = \langle f_j, (T_\sigma)^* g\rangle \longrightarrow \langle f, (T_\sigma)^* g\rangle = \langle T_\sigma f, g\rangle \quad \forall\, g \in \mathcal{S}(\mathbb{R}^d; X^*).$$

Thus, $T_\sigma f_j$ converges to $T_\sigma f$ in $\mathcal{S}'(\mathbb{R}^d; X^{**})$. So T_σ is continuous on $\mathcal{S}'(\mathbb{R}^d; X^{**})$. □

The pseudo-differential operator defined above has a parallel description in terms of a distribution kernel:

$$T_\sigma f(s) = \int_{\mathbb{R}^d} K(s, s-t) f(t) dt,$$

where K is the inverse Fourier transform of σ with respect to the variable ξ, i.e.

$$K(s,t) = \int_{\mathbb{R}^d} \sigma(s,\xi) e^{2\pi \mathrm{i} t\cdot\xi} d\xi. \tag{2.6}$$

In the sequel, we will focus on the symbols in the class $S^n_{1,\delta}$ with $0 \le \delta \le 1$ and $n \in \mathbb{R}$. Similarly to the classical case (see [12], [22], [46] and [50]), we prove that for any operator-valued symbol $\sigma \in S^n_{1,\delta}$, the corresponding kernel K satisfies the following estimates:

Lemma 2.6. *Let $\sigma \in S^n_{1,\delta}$ and $0 \le \delta \le 1$. Then the kernel $K(s,t)$ in* (2.6) *satisfies*

$$\|D^\gamma_s D^\beta_t K(s,t)\|_{B(X)} \le C_{\gamma,\beta} |t|^{-|\gamma|_1 - |\beta|_1 - d - n}, \quad \forall\, t \in \mathbb{R}^d \setminus \{0\}, \tag{2.7}$$

$$\|D^\gamma_s D^\beta_t K(s,t)\|_{B(X)} \le C_{\gamma,\beta,N} |t|^{-N}, \quad \forall\, N > 0 \;\; \textit{if } |t| > 1. \tag{2.8}$$

Proof. This lemma can be deduced easily from the corresponding scalar-valued results, which can be found in many classical works on pseudo-differential operators, for instance, [49, Lemma 5.1.6]. Given $x \in X$ and $x^* \in X^*$ with norms equal to one, it is clear that $\langle x^*, \sigma(s,t)x\rangle$ is a scalar-valued symbol in $S^n_{1,\delta}$, with distribution kernel $\langle x^*, K(s,t)x\rangle$. Thus, we have

$$\langle x^*, D^\gamma_s D^\beta_t K(s,t)x\rangle = D^\gamma_s D^\beta_t [\langle x^*, K(s,t)x\rangle] \le C_{\gamma,\beta} |t|^{-|\gamma|_1 - |\beta|_1 - d - n}, \quad \forall\, t \in \mathbb{R}^d \setminus \{0\}$$

and

$$\langle x^*, D^\gamma_s D^\beta_t K(s,t)x\rangle = D^\gamma_s D^\beta_t [\langle x^*, K(s,t)x\rangle] \le C_{\gamma,\beta,N} |t|^{-N}, \quad \forall\, N > 0 \text{ if } |t| > 1.$$

Then, taking the supremum over x and x^* in the above two inequalities, we get the desired assertion. □

In the classical case, the proof the above lemma makes use of the decomposition of the symbol σ into dyadic pieces. Let $(\widehat{\varphi}_k)_{k\ge 0}$ be the resolution of the unit satisfying (1.3). Set

$$\sigma_k(s,\xi) = \sigma(s,\xi)\widehat{\varphi}_k(\xi), \quad \forall\, (s,\xi) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{2.9}$$

By a similar argument as in the above proof, we also have the following estimates of the corresponding kernels of these pieces σ_k's.

Lemma 2.7. *Let $\sigma \in S^n_{1,\delta}$ and σ_k be as in* (2.9) *and*

$$K_k(s,t) := \int_{\mathbb{R}^d} \sigma_k(s,\xi) e^{2\pi \mathrm{i} t\cdot\xi} d\xi.$$

Then

$$\|D^\gamma_s D^\beta_t K_k(s,t)\|_{B(X)} \lesssim |t|^{-2M} 2^{k(|\beta|_1 + |\gamma|_1 + d - 2M + n)}, \quad \forall\, M \in \mathbb{N}_0.$$

Now we study the composition of pseudo-differential operators. The following proposition gives a rule of the composition of two pseudo-differential operators. Different from the proof of Lemma 2.6, we can not reduce that of the following proposition to the scalar-valued case. So we need to perform an argument which is similar to the classical case. We refer the reader to [45, Theorem 2.5.1] or [46, p. 237], where the case $\delta = 0$ is dealt with in the classical setting. However, for the case $0 < \delta < 1$, the remainder of the Taylor expansion of σ_1 is much harder to handle, which requires a subtler expansion of σ_1.

Proposition 2.8. *Let $0 \le \delta < 1$ and σ_1, σ_2 be two symbols in $S^{n_1}_{1,\delta}$ and $S^{n_2}_{1,\delta}$ respectively. There exists a symbol σ_3 in $S^{n_1+n_2}_{1,\delta}$ such that*

$$T_{\sigma_3} = T_{\sigma_1} T_{\sigma_2}.$$

Moreover,

$$\sigma_3 - \sum_{|\gamma|_1 < N_0} \frac{(2\pi \mathrm{i})^{-|\gamma|_1}}{\gamma!} D^\gamma_\xi \sigma_1 D^\gamma_s \sigma_2 \in S^{n_1+n_2-(1-\delta)N_0}_{1,\delta}, \quad \forall\, N_0 \ge 0. \tag{2.10}$$

Proof. Firstly, we assume that σ_1 and σ_2 have compact supports, so we can use (2.2) as an alternate definition of T_{σ_1} and T_{σ_2}. In this way, $T_{\sigma_1}T_{\sigma_2}$ can be written as follows:

$$T_{\sigma_1}(T_{\sigma_2}f)(s) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_3(s,\xi)f(r)e^{2\pi \mathrm{i}(s-r)\cdot\xi}drd\xi,$$

where

$$\begin{aligned}\sigma_3(s,\xi) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_1(s,\eta)\sigma_2(t,\xi)e^{2\pi \mathrm{i}(s-t)\cdot(\eta-\xi)}dtd\eta \\ &= \int_{\mathbb{R}^d} \sigma_1(s,\xi+\eta)\widehat{\sigma}_2(\eta,\xi)e^{-2\pi \mathrm{i}s\cdot\eta}d\eta\end{aligned} \tag{2.11}$$

with $\widehat{\sigma}_2$ the Fourier transform of σ_2 with respect to the first variable. We expand $\sigma_1(s,\xi+\eta)$ by the Taylor formula:

$$\sigma_1(s,\xi+\eta) = \sum_{|\gamma|_1<N_0}\frac{1}{\gamma!}D_\xi^\gamma\sigma_1(s,\xi)\eta^\gamma + \sum_{N_0\le|\gamma|_1<N}\frac{1}{\gamma!}D_\xi^\gamma\sigma_1(s,\xi)\eta^\gamma + R_N(s,\xi,\eta),$$

with the remainder

$$R_N(s,\xi,\eta) = \sum_{|\gamma|_1=N}\frac{1}{\gamma!}\int_0^1 D_\xi^\gamma\sigma_1(s,\xi+\theta\eta)(1-\theta)^N\eta^\gamma d\theta.$$

Now we replace $\sigma_1(s,\xi+\eta)$ in (2.11) by the above Taylor polynomial and remainder. Notice that

$$\frac{1}{\gamma!}\int_{\mathbb{R}^d} D_\xi^\gamma\sigma_1(s,\xi)\eta^\gamma\widehat{\sigma}_2(\eta,\xi)e^{-2\pi \mathrm{i}s\cdot\eta}d\eta = \frac{(2\pi \mathrm{i})^{-|\gamma|_1}}{\gamma!}D_\xi^\gamma\sigma_1(s,\xi)D_s^\gamma\sigma_2(s,\xi).$$

Thus,

$$\begin{aligned}\sigma_3(s,\xi) = &\Big(\sum_{|\gamma|_1<N_0} + \sum_{N_0\le|\gamma|_1<N}\Big)\frac{(2\pi \mathrm{i})^{-|\gamma|_1}}{\gamma!}D_\xi^\gamma\sigma_1(s,\xi)D_s^\gamma\sigma_2(s,\xi) \\ &+ \int_{\mathbb{R}^d} R_N(s,\xi,\eta)\widehat{\sigma}_2(\eta,\xi)e^{-2\pi \mathrm{i}s\cdot\eta}d\eta.\end{aligned} \tag{2.12}$$

For every γ, the term $D_\xi^\gamma\sigma_1(s,\xi)D_s^\gamma\sigma_2(s,\xi)$ is a symbol in $S_{1,\delta}^{n_1+n_2-(1-\delta)|\gamma|_1}$. Indeed, it is clear that

$$\|D_\xi^\gamma\sigma_1(s,\xi)D_s^\gamma\sigma_2(s,\xi)\|_{B(X)} \lesssim (1+|\xi|)^{n_1-|\gamma|_1}(1+|\xi|)^{n_2+\delta|\gamma|_1} = (1+|\xi|)^{n_1+n_2-(1-\delta)|\gamma|_1}.$$

Moreover, for any $\beta_1,\beta_2\in\mathbb{N}_0^d$, we have $D_s^{\beta_1}\sigma_1\in S_{1,\delta}^{n_1+\delta|\beta_1|_1}, D_s^{\beta_2}\sigma_2\in S_{1,\delta}^{n_2+\delta|\beta_1|_1}$ and $D_\xi^{\beta_1}\sigma_1\in S_{1,\delta}^{n_1-|\beta_2|_1}, D_\xi^{\beta_2}\sigma_2\in S_{1,\delta}^{n_2-|\beta_2|_1}$. Thus, we get

$$\begin{aligned}\big\|D_s^\beta[D_\xi^\gamma\sigma_1(s,\xi)D_s^\gamma\sigma_2(s,\xi)]\big\|_{B(X)} &\lesssim \sum_{\beta_1+\beta_2=\beta}\big\|D_s^{\beta_1}D_\xi^\gamma\sigma_1(s,\xi)D_s^{\beta_2}D_s^\gamma\sigma_2(s,\xi)\big\|_{B(X)} \\ &\lesssim (1+|\xi|)^{n_1+n_2-(1-\delta)|\gamma|_1+\delta|\beta|_1},\end{aligned}$$

and

$$\begin{aligned}\big\|D_\xi^\beta[D_\xi^\gamma\sigma_1(s,\xi)D_s^\gamma\sigma_2(s,\xi)]\big\|_{B(X)} &\lesssim \sum_{\beta_1+\beta_2=\beta}\big\|D_\xi^{\gamma+\beta_1}\sigma_1(s,\xi)D_s^\gamma D_\xi^{\beta_2}\sigma_2(s,\xi)\big\|_{B(X)} \\ &\lesssim (1+|\xi|)^{n_1+n_2-(1-\delta)|\gamma|_1-|\beta|_1}.\end{aligned}$$

By the above estimates, we see that when $N_0\le|\gamma|_1<N$, $D_\xi^\gamma\sigma_1(s,\xi)D_s^\gamma\sigma_2(s,\xi)\in S_{1,\delta}^{n_1+n_2-(1-\delta)N_0}$.

Now we have to treat the last term in (2.12). For the remainder $R_N(s,\xi,\eta)$, we easily check that for any $|\gamma|_1=N$ and $0\le\theta\le1$,

$$\|D_\xi^\gamma\sigma_1(s,\xi+\theta\eta)\|_{B(X)} \le C_N(1+|\xi|)^{n_1-N}, \quad \text{if } |\xi|\ge2|\eta|, \tag{2.13}$$

and

$$\|D_\xi^\gamma\sigma_1(s,\xi+\theta\eta)\|_{B(X)} \le C_N', \quad \forall\,\eta,\xi\in\mathbb{R}^d. \tag{2.14}$$

For $\widehat{\sigma}_2$, by integration by parts, we see that for any $\beta \in \mathbb{N}_0^d$ such that $|\beta|_1 = \widetilde{N}$, we have

$$\begin{aligned}(-2\pi \mathrm{i}\eta)^\beta \widehat{\sigma}_2(\eta,\xi) &= \int_{\mathbb{R}^d} (-2\pi \mathrm{i}\eta)^\beta e^{-2\pi \mathrm{i} t\cdot\eta}\sigma_2(t,\xi)dt \\ &= \int_{\mathbb{R}^d} D_t^\beta (e^{-2\pi \mathrm{i} t\cdot\eta})\sigma_2(t,\xi)dt \\ &= (-1)^\beta \int_{\mathbb{R}^d} e^{-2\pi \mathrm{i} t\cdot\eta} D_t^\beta \sigma_2(t,\xi)dt.\end{aligned}$$

Denote the compact t-support of $\sigma_2(t,\xi)$ by Ω. Then the above calculation immediately implies that

$$\|\widehat{\sigma}_2(\eta,\xi)\|_{B(X)} \lesssim |\Omega|(1+|\eta|)^{-\widetilde{N}}(1+|\xi|)^{n_2+\delta\widetilde{N}}. \tag{2.15}$$

We keep the constant $|\Omega|$ in this inequality for the moment, and will see in the next step that our final result does not depend on the volume of this support. Take $\widetilde{N}$ large enough so that

$$\widetilde{N} > \max\Big\{\frac{d}{1-\widetilde{\delta}}, \frac{(1-\delta)N_0}{\widetilde{\delta}-\delta}, \frac{d-n_1+(1-\delta)N_0}{1-2\delta}\Big\},$$

and take $N = \widetilde{\delta}\widetilde{N}$ with $0 \le \delta < \widetilde{\delta} < 1$. Continuing the estimate of the last term in (2.12), inequalities (2.13) and (2.15) give

$$\begin{aligned}&\Big\| \int_{|\eta|\le\frac{|\xi|}{2}} \int_0^1 D_\xi^\gamma \sigma_1(s,\xi+\theta\eta)(1-\theta)^N \eta^\gamma \widehat{\sigma}_2(\eta,\xi) e^{-2\pi \mathrm{i} s\cdot\eta} d\theta d\eta \Big\|_{B(X)} \\ &\lesssim \int_{\mathbb{R}^d} |\eta|^N (1+|\eta|)^{-\widetilde{N}} d\eta \cdot (1+|\xi|)^{n_1+n_2-N+\delta\widetilde{N}} \\ &\le \int_{\mathbb{R}^d} (1+|\eta|)^{(\widetilde{\delta}-1)\widetilde{N}} d\eta \cdot (1+|\xi|)^{n_1+n_2+(\delta-\widetilde{\delta})\widetilde{N}} \\ &\lesssim (1+|\xi|)^{n_1+n_2+(\delta-\widetilde{\delta})\widetilde{N}}.\end{aligned}$$

Moreover, since $\widetilde{N} \ge \frac{(1-\delta)N_0}{\widetilde{\delta}-\delta}$, we have

$$(1+|\xi|)^{n_1+n_2+(\delta-\widetilde{\delta})\widetilde{N}} \le (1+|\xi|)^{n_1+n_2-(1-\delta)N_0}.$$

According to (2.14) and (2.15), we get

$$\begin{aligned}&\Big\| \int_{|\eta|>\frac{|\xi|}{2}} \int_0^1 D_\xi^\gamma \sigma_1(s,\xi+\theta\eta)(1-\theta)^N \eta^\gamma \widehat{\sigma}_2(\eta,\xi) e^{-2\pi \mathrm{i} s\cdot\eta} d\theta d\eta \Big\|_{B(X)} \\ &\lesssim \int_{|\eta|>\frac{|\xi|}{2}} |\eta|^N (1+|\eta|)^{-\widetilde{N}} d\eta \cdot (1+|\xi|)^{n_2+\delta\widetilde{N}} \\ &\lesssim (1+|\xi|)^{n_2+N+d-(1-\delta)\widetilde{N}} \le (1+|\xi|)^{n_1+n_2-(1-\delta)N_0}.\end{aligned}$$

Therefore, $R_N(s,\xi,\eta) \in S_{1,\delta}^{n_1+n_2-(1-\delta)N_0}$. Combining the estimates above, we see that, if we set $R_{N_0}(s,\xi,\eta) = \sum_{N_0\le|\gamma|_1<N} \frac{1}{\gamma!} D_\xi^\gamma \sigma_1(s,\xi)\eta^\gamma + R_N(s,\xi,\eta)$, then $R_{N_0}(s,\xi,\eta) \in S_{1,\delta}^{n_1+n_2-(1-\delta)N_0}$. This proves the assertion (2.10) when σ_2 has compact support with respect to the first variable.

Noticing that the above proof depends on the constant $|\Omega|$ in (2.15), we now make use of the resolution of the unit in (1.4) to deal with general symbol σ_2 with arbitrary s-support. For each $k \in \mathbb{Z}^d$, denote $\sigma_{2,k}(s,\xi) = \mathcal{X}_k(s)\sigma_2(s,\xi)$ and

$$\sigma_{3,k}(s,\xi) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_1(s,\eta)\sigma_{2,k}(t,\xi)e^{2\pi \mathrm{i}(s-t)\cdot(\eta-\xi)}dtd\eta\,.$$

It has already been established that

$$\sigma_{3,k} - \sum_{|\gamma|_1<N_0} \frac{(2\pi \mathrm{i})^{-|\gamma|_1}}{\gamma!} D_\xi^\gamma \sigma_1 D_s^\gamma \sigma_{2,k} \in S_{1,\delta}^{n_1+n_2-(1-\delta)N_0}, \quad \forall\, N_0>0, k\in\mathbb{Z}^d, \tag{2.16}$$

with relevant constants uniform in k. Observe that if two symbols b_1, b_2 in some $S_{1,\delta}^n$ have disjoint s-supports, with

$$\|D_s^\gamma D_\xi^\beta b_i(s,\xi)\|_{B(X)} \le C_{i,\gamma,\beta}(1+|\xi|)^{n+\delta|\gamma|_1-|\beta|_1}, \quad i=1,2,$$

then $b_1 + b_2 \in S^n_{1,\delta}$ with

$$\|D_s^\gamma D_\xi^\beta\big(b_1(s,\xi)+b_2(s,\xi)\big)\|_{B(X)} \le \max\{C_{1,\gamma,\beta}, C_{2,\gamma,\beta}\}(1+|\xi|)^{n+\delta|\gamma|_1-|\beta|_1}.$$

For our use, we construct a partition of $\mathbb{Z}^d$ with subsets $U_1, U_2, \cdots, U_{2^d}$ such that for any k_1, k_2 in each U_j, the supports $\operatorname{supp}\mathcal{X}_{k_1}$ and $\operatorname{supp}\mathcal{X}_{k_2}$ are disjoint. More precisely, let π: $\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ be the canonical projection sending even integer to 0 and odd integer to 1. Let π^d: $\mathbb{Z}^d \longrightarrow (\mathbb{Z}/2\mathbb{Z})^d$ be the d-fold product of π. Then $(U_j)_{j\in(\mathbb{Z}/2\mathbb{Z})^d} = \big((\pi^d)^{-1}(j)\big)_{j\in(\mathbb{Z}/2\mathbb{Z})^d}$ gives the desired partition of $\mathbb{Z}^d$. Summing over (2.16) in each U_j, we get a symbol still in $S^{n_1+n_2-(1-\delta)N_0}_{1,\delta}$, that is,

$$\sum_{k\in U_j}\sigma_{3,k} - \sum_{k\in U_j}\sum_{|\gamma|_1<N_0}\frac{(2\pi\mathrm{i})^{-|\gamma|_1}}{\gamma!}D_\xi^\gamma\sigma_1 D_s^\gamma\sigma_{2,k} \in S^{n_1+n_2-(1-\delta)N_0}_{1,\delta}.$$

Taking the finite sum over $\{U_j\}_{1\le j\le 2^d}$, we get the asymptotic formula (2.10) in this case.

Finally, let us get rid of the additional assumption that σ_1 and σ_2 have compact supports. We define σ_3^j as follows:

$$T_{\sigma_3^j} = T_{\sigma_1^j}T_{\sigma_2^j}.$$

where $\sigma_1^j(s,\xi) = \sigma_1(s,\xi)\eta(2^{-j}s, 2^{-j}\xi)$ and $\sigma_2^j(s,\xi) = \sigma_2(s,\xi)\eta(2^{-j}s,2^{-j}\xi)$ with η given in (2.3). Notice that the σ_1^j's and the σ_2^j's are in the class $S^{n_1}_{1,\delta}$ and $S^{n_2}_{1,\delta}$ respectively with symbolic constants uniform in j. Therefore, the above arguments ensure that σ_3^j belongs to $S^{n_1+n_2}_{1,\delta}$ and satisfies (2.10) uniformly in j. Passing to the limit, we get that $\sigma_3 \in S^{n_1+n_2}_{1,\delta}$ and satisfies (2.10). Furthermore, by (2.4), we get

$$T_{\sigma_3} = T_{\sigma_1}T_{\sigma_2}.$$

The proof is complete. $\square$

By a similar argument as the above proof, we also get the asymptotic formula for the adjoint of a pseudo-differential operator with symbol in the class $S^n_{1,\delta}$ when $0 \le \delta < 1$.

Proposition 2.9. *Let $0\le\delta<1$, $n\in\mathbb{R}$ and σ be a symbol in $S^n_{1,\delta}$. There exists a symbol $\widetilde{\sigma} \in S^n_{1,\delta}$ such that $T_{\widetilde{\sigma}} = (T_\sigma)^*$. Moreover,*

$$\widetilde{\sigma} - \sum_{|\gamma|_1<N_0}\frac{(2\pi\mathrm{i})^{-|\gamma|_1}}{\gamma!}D_\xi^\gamma D_s^\gamma\sigma^* \in S^{n-(1-\delta)N_0}_{1,\delta}, \quad \forall\, N_0 \ge 0.$$

Proof. By (2.5), we get the formal expression of $\widetilde{\sigma}$ that

$$\begin{aligned}\widetilde{\sigma}(s,\xi) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sigma^*(t,\eta)e^{2\pi\mathrm{i}(s-t)\cdot(\eta-\xi)}dtd\eta\\ &= \int_{\mathbb{R}^d}\widehat{\sigma}^*(\eta,\xi+\eta)e^{2\pi\mathrm{i}s\cdot\eta}d\eta,\end{aligned}$$

where $\widehat{\sigma}^*$ is the Fourier transform of σ^* with respect to the first variable. By the same argument used in the proof of the previous proposition, we may focus on the symbol with compact t-support. Taking the Taylor expression of $\widehat{\sigma}^*(\eta,\xi+\eta)$, we get

$$\widehat{\sigma}^*(\eta,\xi+\eta) = \sum_{|\gamma|_1<N_0}\frac{1}{\gamma!}D_\xi^\gamma\widehat{\sigma}^*(\eta,\xi)\eta^\gamma + \sum_{N_0\le|\gamma|_1<N}\frac{1}{\gamma!}D_\xi^\gamma\widehat{\sigma}^*(\eta,\xi)\eta^\gamma + R_N(\xi,\eta).$$

As before, we can show that

$$\frac{1}{\gamma!}\int_{\mathbb{R}^d}D_\xi^\gamma\widehat{\sigma}^*(\eta,\xi)\eta^\gamma e^{2\pi\mathrm{i}s\cdot\eta}d\eta = \frac{(2\pi\mathrm{i})^{-|\gamma|_1}}{\gamma!}D_\xi^\gamma D_s^\gamma\widehat{\sigma}^*(s,\xi) \in S^{n-(1-\delta)|\gamma|_1}_{1,\delta}.$$

On the other hand, we can also show that

$$\Big\|\int_{\mathbb{R}^d}R_N(\xi,\eta)e^{2\pi\mathrm{i}s\cdot\eta}d\eta\Big\|_{B(X)} \lesssim (1+|\xi|)^{n-(1-\delta)N_0}$$

by splitting the integral over η into two parts. Moreover, repeating the above procedure to its derivatives, we have $\int_{\mathbb{R}^d}R_N(\xi,\eta)e^{2\pi\mathrm{i}s\cdot\eta}d\eta \in S^{n-(1-\delta)N_0}_{1,\delta}$. Thus, the proposition is proved. $\square$

Remark 2.10. The above two propositions show that the symbol class $S^0_{1,\delta}$ is closed under the product and adjoint of pseudo-differential operators. This is one of the reasons why we call symbols in $S^0_{1,\delta}$ with $0 \le \delta < 1$ *regular symbols*; respectively, we call symbols in $S^0_{1,1}$ *forbidden symbols.* In the next section, we will see the different behaviours of regular and forbidden symbols on L_2 spaces, that also distinguish these two kinds of symbols.

3. Mapping properties on Sobolev and Besov spaces

In the sequel, we will mainly consider pseudo-differential operators whose symbols take values in some von Neumann algebra $\mathcal{M}$. If we take $X = L_1(\mathcal{M}) + \mathcal{M}$, then $\mathcal{M}$ admits an isometric embedding into $B(X)$ by left multiplication. In this way, these $\mathcal{M}$-valued symbols can be seen as a special case of the $B(X)$-valued symbols defined in the previous section. On the other hand, if we embed $\mathcal{M}$ into $B(X)$ by right multiplication, we get another kind of $\mathcal{M}$-valued symbol actions. Accordingly, we define

$$T^c_\sigma f(s) = \int_{\mathbb{R}^d} \sigma(s,\xi)\widehat{f}(\xi)e^{2\pi \mathrm{i} s\cdot\xi}d\xi$$

and

$$T^r_\sigma f(s) = \int_{\mathbb{R}^d} \widehat{f}(\xi)\sigma(s,\xi)e^{2\pi \mathrm{i} s\cdot\xi}d\xi.$$

All the conclusions proved in the previous section still hold for both T^c_σ and T^r_σ in parallel. In the following sections, we mainly focus on the operators T^c_σ.

This section is devoted to the study of the continuity of operator-valued pseudo-differential operators on Sobolev and Besov spaces. Let us now give some background on these function spaces.

For $\alpha \in \mathbb{R}, 1 \le p \le \infty$ and a Banach space X, the potential Sobolev space $H^\alpha_p(\mathbb{R}^d;X)$ is the space of all distributions in $S'(\mathbb{R}^d;X)$ which have finite Sobolev norm $\|f\|_{H^\alpha_p} = \|J^\alpha f\|_{L_p(\mathbb{R}^d;X)}$. It is well known that the potential Sobolev spaces are closely related to Besov spaces. We still use the resolution of the unit $(\varphi_k)_{k\ge 0}$ introduced in (1.3) to define Besov spaces. Given $\alpha \in \mathbb{R}^d$ and $1 \le p,q \le \infty$, the Besov space $B^\alpha_{p,q}(\mathbb{R}^d;X)$ is defined to be the subspace of $S'(\mathbb{R}^d;X)$ consisting of all f such that

$$\|f\|_{B^\alpha_{p,q}} = \Big(\sum_{k\ge 0} 2^{qk\alpha}\|\varphi_k * f\|^q_{L_p(\mathbb{R}^d;X)}\Big)^{\frac{1}{q}} < \infty.$$

The above vector-valued Besov spaces $B^\alpha_{p,q}(\mathbb{R}^d;X)$ have been studied by many authors, see for instance [1].

Instead of the Banach-valued spaces defined above, we prefer to study the operator-valued spaces $H^\alpha_p(\mathbb{R}^d;L_p(\mathcal{M}))$ and $B^\alpha_{p,q}(\mathbb{R}^d;L_p(\mathcal{M}))$. Obviously, the main difference is that the Banach space X varies for different p. The following inclusions are easy to check for every $1 \le p \le \infty$,

$$B^\alpha_{p,1}(\mathbb{R}^d;L_p(\mathcal{M})) \subset H^\alpha_p(\mathbb{R}^d;L_p(\mathcal{M})) \subset B^\alpha_{p,\infty}(\mathbb{R}^d;L_p(\mathcal{M})).$$

Besov spaces are stable under real interpolation. More precisely, if $\alpha_0,\alpha_1 \in \mathbb{R}$, $\alpha_0 \ne \alpha_1$ and $0 < \theta < 1$, then

$$\big(B^{\alpha_0}_{p,q_0}(\mathbb{R}^d;L_p(\mathcal{M})), B^{\alpha_1}_{p,q_1}(\mathbb{R}^d;L_p(\mathcal{M}))\big)_{\theta,q} = B^\alpha_{p,q}(\mathbb{R}^d;L_p(\mathcal{M})), \tag{3.1}$$

for $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1,\ p,q,q_0,q_1 \in [1,\infty]$. This result is a particular case of Amann's Banach-valued counterpart in [1] with $X = L_p(\mathcal{M})$, which can be deduced from [6, Theorem 5.6.1] by considering the pairing between $\ell^{\alpha_0}_{q_0}(L_p(\mathbb{R}^d;X))$ and $\ell^{\alpha_1}_{q_1}(L_p(\mathbb{R}^d;X))$.

3.1. **Mapping properties on L_2-Sobolev spaces.**

We start by presenting an L_2-theorem. It is a noncommutative analogue of the corresponding classical theorem, which can be found in many works, for instance, [46, 47, 45]. We will work on the exotic class $S^0_{\delta,\delta}$ with $0 \le \delta < 1$, since we have the inclusion $S^0_{1,\delta} \subset S^0_{\delta,\delta}$. Our argument adapts [46, Proposition VII.2.4, Theorem VII.2.5] to the operator-valued case. The Cotlar-Stein Almost Orthogonality Lemma [15, 46] plays a crucial role in our proof. Namely, given a family of operators $(T_j)_j \subset B(H)$ with H a Hilbert space, and a positive sequence $\big(c(j)\big)_j$ such that $\sum_j c(j) = C < \infty$, if the T_j's satisfy:

$$\|T^*_k T_j\|_{B(H)} \le |c(k-j)|^2$$

and

$$\|T_k T_j^*\|_{B(H)} \le |c(k-j)|^2,$$

then we have

$$\|\sum_j T_j\|_{B(H)} \le C.$$

Lemma 3.1. *Assume $\sigma \in S^0_{0,0}$. Then T^c_σ is bounded on $L_2(\mathcal{N})$.*

Proof. By the Plancherel formula, it is enough to prove the $L_2(\mathcal{N})$-boundedness of the following operator:

$$S^c_\sigma(f)(s) = \int_{\mathbb{R}^d} \sigma(s,\xi) f(\xi) e^{2\pi \mathrm{i} s\cdot\xi} d\xi.$$

Let us make use of the resolution of the unit $(\mathcal{X}_k)_{k\in\mathbb{Z}^d}$ introduced in (1.4) to decompose S^c_σ into almost orthogonal pieces. Denote $\mathrm{k} = (k,k') \in \mathbb{Z}^d \times \mathbb{Z}^d$, and set

$$\sigma_{\mathrm{k}}(s,\xi) = \mathcal{X}_k(s)\sigma(s,\xi)\mathcal{X}_{k'}(\xi),$$

Then, the series $\sum_{\mathrm{k}\in\mathbb{Z}^d\times\mathbb{Z}^d} S^c_{\sigma_{\mathrm{k}}}$ converges in the strong operator topology and

$$S^c_\sigma = \sum_{\mathrm{k}\in\mathbb{Z}^d\times\mathbb{Z}^d} S^c_{\sigma_{\mathrm{k}}}.$$

We claim that $(S^c_{\sigma_{\mathrm{k}}})_{\mathrm{k}}$ satisfies the almost-orthogonality estimates, i.e., for any $N\in\mathbb{N}$,

$$\|(S^c_{\sigma_{\mathrm{k}}})^* S^c_{\sigma_{\mathrm{j}}}\|_{B(L_2(\mathcal{N}))} \le C_N(1+|\mathrm{k}-\mathrm{j}|)^{-2N},$$

and

$$\|S^c_{\sigma_{\mathrm{k}}}(S^c_{\sigma_{\mathrm{j}}})^*\|_{B(L_2(\mathcal{N}))} \le C_N(1+|\mathrm{k}-\mathrm{j}|)^{-2N},$$

where the constant C_N is independent of $\mathrm{k}=(k,k')$ and $\mathrm{j}=(j,j')$. Armed with this claim, we can then apply the Cotlar-Stein Almost Orthogonality Lemma stated previously to the operators $(S^c_{\sigma_{\mathrm{k}}})_{\mathrm{k}}$ with $c(\mathrm{j}) = (1+|\mathrm{j}|)^{-N}$, $N>2d$. Then, we will have

$$\|S^c_\sigma\|_{B(L_2(\mathcal{N}))} = \|\sum_{\mathrm{k}\in\mathbb{Z}^d\times\mathbb{Z}^d} S^c_{\sigma_{\mathrm{k}}}\|_{B(L_2(\mathcal{N}))} \le C.$$

Now we prove the claim. Note that for any $f\in L_2(\mathcal{N})$,

$$(S^c_{\sigma_{\mathrm{k}}})^* S^c_{\sigma_{\mathrm{j}}}(f)(\xi) = \int_{\mathbb{R}^d} \sigma_{\mathrm{k},\mathrm{j}}(\xi,\eta) f(\eta) d\eta,$$

where

$$\sigma_{\mathrm{k},\mathrm{j}}(\xi,\eta) = \int_{\mathbb{R}^d} \sigma^*_{\mathrm{k}}(s,\xi)\sigma_{\mathrm{j}}(s,\eta) e^{2\pi \mathrm{i} s\cdot(\eta-\xi)} ds. \tag{3.2}$$

By the definition of σ_{k}, we see that if $k-j\notin 2Q_{0,0}$ (recalling that $Q_{0,0}$ is the unit cube centered at the origin), σ_{k} and σ_{j} have disjoint s-support, so

$$\sigma^*_{\mathrm{k}}\sigma_{\mathrm{j}} = 0.$$

When $k-j\in 2Q_{0,0}$, using the identity

$$(1-\Delta_s)^N e^{2\pi \mathrm{i} s\cdot(\eta-\xi)} = (1+4\pi^2|\eta-\xi|^2)^N e^{2\pi \mathrm{i} s\cdot(\eta-\xi)},$$

we integrate (3.2) by parts, which gives

$$\|\sigma_{\mathrm{k},\mathrm{j}}(\xi,\eta)\|_{\mathcal{M}} \le C_N \mathcal{X}_{k'}(\xi)\mathcal{X}_{j'}(\eta)(1+|\xi-\eta|)^{-2N}.$$

Whence,

$$\max\Big\{\int_{\mathbb{R}^d}\|\sigma_{\mathrm{k},\mathrm{j}}(\xi,\eta)\|_{\mathcal{M}} d\xi, \int_{\mathbb{R}^d}\|\sigma_{\mathrm{k},\mathrm{j}}(\xi,\eta)\|_{\mathcal{M}} d\eta\Big\} \le C'_N(1+|\mathrm{k}-\mathrm{j}|)^{-2N}. \tag{3.3}$$

For any $f\in L_2(\mathcal{N})$, there exists $g\in L_2(\mathcal{N})$ with norm one such that

$$\big\|(S^c_{\sigma_{\mathrm{k}}})^* S^c_{\sigma_{\mathrm{j}}} f\big\|_{L_2(\mathcal{N})} = \Big|\tau\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sigma_{\mathrm{k},\mathrm{j}}(\xi,\eta) f(\eta)\, d\eta\, g^*(\xi)\, d\xi\Big|.$$

Applying the Hölder inequality and (3.3), we get

$$\begin{aligned}
&\Big|\tau\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sigma_{\mathrm{k,j}}(\xi,\eta)f(\eta)\,d\eta\,g^*(\xi)\,d\xi\Big| \\
&\leq \big(\tau\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\|\sigma_{\mathrm{k,j}}(\xi,\eta)\|_{\mathcal{M}}|f(\eta)|^2d\eta d\xi\big)^{\frac{1}{2}}\big(\tau\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\|\sigma_{\mathrm{k,j}}(\xi,\eta)\|_{\mathcal{M}}|g(\xi)|^2d\xi d\eta\big)^{\frac{1}{2}} \\
&\leq C'_N(1+|\mathrm{k}-\mathrm{j}|)^{-2N}\|f\|_{L_2(\mathcal{N})}.
\end{aligned}$$

Thus, $\|(S^c_{\sigma_\mathrm{k}})^*S^c_{\sigma_\mathrm{j}}\|_{B(L_2(\mathcal{N}))}\leq C'_N(1+|\mathrm{k}-\mathrm{j}|)^{-2N}$. On the other hand, a similar argument also shows that

$$\|S^c_{\sigma_\mathrm{k}}(S^c_{\sigma_\mathrm{j}})^*\|_{B(L_2(\mathcal{N}))}\leq C'_N(1+|\mathrm{k}-\mathrm{j}|)^{-2N},$$

which proves the claim. □

A weak form of Cotlar-Stein's almost orthogonality lemma also plays a crucial role. As before, we suppose that $\sum_j c(j)=C<\infty$. This time we assume that the T_j's satisfy:

$$\sup_j\|T_j\|_{B(H)}\leq C \tag{3.4}$$

and the following conditions hold for $j\neq k$:

$$\|T_jT_k^*\|_{B(H)}=0\quad\text{and}\quad\|T_j^*T_k\|_{B(H)}\leq c(j)c(k). \tag{3.5}$$

Then we have

$$\big\|\sum_j T_j\big\|_{B(H)}\leq\sqrt{2}C.$$

Proposition 3.2. *Let $\sigma\in S^0_{\delta,\delta}$ with $0\leq\delta<1$. Then T^c_σ is bounded on $L_2(\mathcal{N})$.*

Proof. To prove this lemma, we apply Cotlar's lemma as stated above. Let $(\widehat{\varphi}_j)_{j\geq 0}$ be the resolution of the unit defined in (1.3). We can decompose T^c_σ as follows:

$$T^c_\sigma=\sum_{j=0}^{\infty}T^c_{\sigma_j}=\sum_{j\text{ even}}T^c_{\sigma_j}+\sum_{j\text{ odd}}T^c_{\sigma_j},$$

where $\sigma_j(s,\xi)=\widehat{\varphi}_j(\xi)\sigma(s,\xi)$. Note that the symbols in either odd or even summand have disjoint ξ-supports. We will only treat the odd part, since the other part can be dealt with in a similar way. It is clear that $T^c_{\sigma_j}(T^c_{\sigma_k})^*=0$ if $j\neq k$, since $T^c_{\sigma_j}(T^c_{\sigma_k})^*=T^c_\sigma M_{\widehat{\varphi}_j}M_{\overline{\widehat{\varphi}}_k}(T^c_\sigma)^*$ and $\widehat{\varphi}_j$, $\widehat{\varphi}_k$ have disjoint supports. Now let us estimate the second inequality in (3.5), i.e. the norm of $(T^c_{\sigma_k})^*T^c_{\sigma_j}$. Since

$$(T^c_{\sigma_k})^*(f)(s)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sigma_k^*(t,\xi)f(t)e^{2\pi\mathrm{i}\xi\cdot(s-t)}dtd\xi,$$

and

$$T^c_{\sigma_j}(f)(t)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sigma_j(t,\eta)f(r)e^{2\pi\mathrm{i}\eta\cdot(t-r)}drd\eta.$$

Then we have

$$(T^c_{\sigma_k})^*T^c_{\sigma_j}(f)(s)=\int_{\mathbb{R}^d}K(s,r)f(r)dr,$$

with

$$K(s,r)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sigma_k^*(t,\xi)\sigma_j(t,\eta)e^{2\pi\mathrm{i}[\eta\cdot(t-r)+\xi\cdot(s-t)]}dtd\eta d\xi.$$

Writing

$$e^{2\pi\mathrm{i}(\eta-\xi)\cdot t}=\frac{(1-\Delta_t)^N}{(1+4\pi^2|\xi-\eta|^2)^N}e^{2\pi\mathrm{i}(\eta-\xi)\cdot t},$$

$$e^{2\pi\mathrm{i}(t-r)\cdot\eta}=\frac{(1-\Delta_\eta)^N}{(1+4\pi^2|t-r|^2)^N}e^{2\pi\mathrm{i}(t-r)\cdot\eta},$$

and

$$e^{2\pi\mathrm{i}(s-t)\cdot\xi}=\frac{(1-\Delta_\xi)^N}{(1+4\pi^2|s-t|^2)^N}e^{2\pi\mathrm{i}(s-t)\cdot\xi},$$

we use the integration by parts with respect to the variables t, ξ and η. By standard calculation (see [46, Theorem 2, p. 286] for more details), we get

$$\|K(s,r)\|_{\mathcal{M}} \lesssim 4^{\max(k,j)((\delta-1)N+d)} \int Q(s-t)Q(t-r)dt,$$

where $Q(t) = (1+|t|)^{-2N}$, if $k \neq j$. Denote $K_0(s,r) = \int Q(s-t)Q(t-r)dt$, then

$$\int_{\mathbb{R}^d} K_0(s,r)ds = \int_{\mathbb{R}^d} K_0(s,r)dr = \big(\int_{\mathbb{R}^d} (1+|t|)^{-2N}dt\big)^2 < \infty. \tag{3.6}$$

For any $f \in L_2(\mathcal{N})$, there exists $g \in L_2(\mathcal{N})$ with norm one such that

$$\big\|(T^c_{\sigma_k})^* T^c_{\sigma_j} f\big\|_{L_2(\mathcal{N})} = \Big|\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(s,r)f(r)g(s)drds\Big|.$$

Applying the Hölder inequality and (3.6), we get

$$\begin{aligned}
&\Big|\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(s,r)f(r)g(s)drds\Big| \\
&\leq \big(\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|K(s,r)\|_{\mathcal{M}}|f(r)|^2dsdr\big)^{\frac12}\big(\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|K(s,r)\|_{\mathcal{M}}|g(s)|^2dsdr\big)^{\frac12} \\
&\lesssim 4^{\max(k,j)((\delta-1)N+d)}\|f\|_{L_2(\mathcal{N})},
\end{aligned}$$

which implies that

$$\big\|(T^c_{\sigma_k})^* T^c_{\sigma_j}\big\|_{B(L_2(\mathcal{N}))} \lesssim c(j)c(k), \quad j \neq k,$$

with $c(j) = 2^{j((\delta-1)N+d)}$. If we take $N > \frac{d}{1-\delta}$, the sequence $\big(c(j)\big)_j$ is summable.

In order to apply Cotlar-Stein's lemma, it remains to show that $T^c_{\sigma_j}$'s satisfy (3.4). To this end, we do some technical modifications. Set

$$\widetilde{\sigma}_j = \sigma_j(2^{-j\delta}\cdot, 2^{j\delta}\cdot).$$

We can easily check that the $\widetilde{\sigma}_j$'s belong to $S^0_{0,0}$, uniformly in j. Then, by Lemma 3.1, the $T^c_{\widetilde{\sigma}_j}$'s are bounded on $L_2(\mathcal{N})$ uniformly in j. If Λ_j denotes the dilation operator given by

$$\Lambda_j(f) = f(2^{j\delta}\cdot),$$

then, we can easily verify that

$$T^c_{\sigma_j} = \Lambda_j T^c_{\widetilde{\sigma}_j}\Lambda_j^{-1}.$$

Thus,

$$\|T^c_{\sigma_j}\|_{B(L_2(\mathcal{N}))} \leq \|T^c_{\widetilde{\sigma}_j}\|_{B(L_2(\mathcal{N}))} < \infty.$$

Therefore, $(T^c_{\sigma_j})_{j\geq 0}$ satisfy the assumptions of Cotlar's lemma. So we get

$$\|T^c_\sigma\|_{B(L_2(\mathcal{N}))} = \|\sum_{j=0}^{\infty} T^c_{\sigma_j}\|_{B(L_2(\mathcal{N}))} < \infty.$$

Thus, T^c_σ is bounded on $L_2(\mathcal{N})$. □

By Proposition 2.8 and the fact that $S^0_{1,\delta} \subset S^0_{\delta,\delta}$ for $0 \leq \delta < 1$, we have

Corollary 3.3. *Let $\sigma \in S^0_{1,\delta}$ with $0 \leq \delta < 1$. Then T^c_σ is bounded on $H^\alpha_2(\mathbb{R}^d; L_2(\mathcal{M}))$ for any $\alpha \in \mathbb{R}$.*

Proof. If $\alpha = 0$, by Proposition 3.2 and the inclusion $S^0_{1,\delta} \subset S^0_{\delta,\delta}$, we see the boundedness of T^c_σ on $L_2(\mathbb{R}^d; L_2(\mathcal{M}))$. For general $\alpha \neq 0$, we use the lifting property of $H^\alpha_2(\mathbb{R}^d; L_2(\mathcal{M}))$, which follows easily from the definition. By Proposition 2.8,

$$T^c_{\sigma^\alpha} = J^\alpha T^c_\sigma J^{-\alpha}$$

is still a pseudo-differential operator with symbol σ^α in $S^0_{1,\delta}$. Then for any $f \in H^\alpha_2(\mathbb{R}^d; L_2(\mathcal{M}))$,

$$\|T^c_\sigma f\|_{H^\alpha_2} = \|J^{-\alpha}T^c_{\sigma^\alpha}J^\alpha f\|_{H^\alpha_2} = \|T^c_{\sigma^\alpha}J^\alpha f\|_2 \lesssim \|J^\alpha f\|_2 = \|f\|_{H^\alpha_2}.$$

Therefore, T^c_σ is bounded on $\in H^\alpha_2(\mathbb{R}^d; L_2(\mathcal{M}))$. □

It is well known [46] that there exist symbols in $S^0_{1,1}$ such that the associated pseudo-differential operators are not bounded on $L_2(\mathbb{R}^d)$. Alternatively, the regularity of operators with forbidden symbols on Sobolev spaces $H^\alpha_p(\mathbb{R}^d)$, Besov spaces $B^\alpha_{p,q}(\mathbb{R}^d)$ and Triebel-Lizorkin spaces $F^\alpha_{p,q}(\mathbb{R}^d)$ with $\alpha > 0$ has been widely investigated, see [35, 36, 7, 44, 49].

The following proposition states the regularity of pseudo-differential operators with forbidden symbols on $H^\alpha_2(\mathbb{R}^d; L_2(\mathcal{M}))$, which will be useful when studying that on Triebel-Lizorkin spaces.

Proposition 3.4. *Let $\sigma \in S^0_{1,1}$. Then T^c_σ is bounded on $H^\alpha_2(\mathbb{R}^d; L_2(\mathcal{M}))$ for any $\alpha > 0$.*

Proof. Let $(\varphi_j)_{j\geq 0}$ be the resolution of the unit satisfying (1.3). It is straightforward to show that $H^\alpha_2(\mathbb{R}^d; L_2(\mathcal{M}))$ admits an equivalent norm:

$$\|f\|_{H^\alpha_2(\mathbb{R}^d;L_2(\mathcal{M}))} \approx \big(\sum_{j\geq 0} 2^{2j\alpha}\|\varphi_j * f\|^2_{L_2(\mathcal{N})}\big)^{\frac12} = \|f\|_{B^\alpha_{2,2}(\mathbb{R}^d;L_2(\mathcal{M}))}. \tag{3.7}$$

Thus, it suffices to consider the boundedness of T^c_σ on $B^\alpha_{2,2}(\mathbb{R}^d; L_2(\mathcal{M}))$. Let σ_k with $k \in \mathbb{N}_0$ be the dyadic decomposition of σ given in (2.9). By the support assumptions of $\widehat{\varphi}$ and $\widehat{\varphi}_0$, we have

$$T^c_{\sigma_k}(f) = T^c_{\sigma_k}(f_k),$$

where $f_k = (\varphi_{k-1} + \varphi_k + \varphi_{k+1}) * f$ for $k \geq 1$, and $f_0 = (\varphi_0 + \varphi_1) * f$. Applying Lemma 2.7 to K_k with $M = 0$, we get

$$\int_{|s-t|\leq 2^{-k}} \|D^\gamma_s K_k(s, s-t)\|_{\mathcal{M}} dt \lesssim \int_{|s-t|\leq 2^{-k}} 2^{k(|\gamma|_1+d)} dt \approx 2^{k|\gamma|_1}.$$

If $d+1$ is even, applying Lemma 2.7 again to K_k with $2M = d+1$, we get

$$\int_{|s-t|>2^{-k}} \|D^\gamma_s K_k(s, s-t)\|_{\mathcal{M}} dt \lesssim \int_{|s-t|>2^{-k}} 2^{k(|\gamma|_1-1)}|s-t|^{-d-1} dt \approx 2^{k|\gamma|_1}\ ;$$

if $d+2$ is even, letting $2M = d+2$ in Lemma 2.7, we get the same estimate. Therefore, summing up the above estimates of $\int_{|s-t|\leq 2^{-k}}$ and $\int_{|s-t|>2^{-k}}$, we obtain

$$\int_{\mathbb{R}^d} \|D^\gamma_s K_k(s, s-t)\|_{\mathcal{M}} dt \lesssim 2^{k|\gamma|_1}. \tag{3.8}$$

Since the estimate of $\|D^\gamma_s K_k(s, s-t)\|_{\mathcal{M}}$ is symmetric in s and t, the same proof also shows that

$$\int_{\mathbb{R}^d} \|D^\gamma_s K_k(s, s-t)\|_{\mathcal{M}} ds \lesssim 2^{k|\gamma|_1}. \tag{3.9}$$

For any $f \in H^\alpha_2(\mathbb{R}^d; L_2(\mathcal{M}))$ and $k \in \mathbb{N}_0$, there exists $g_k \in L_2(\mathcal{N})$ with norm one such that $\|D^\gamma_s T^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})} = \big|\tau \int_{\mathbb{R}^d} D^\gamma_s T^c_{\sigma_k}(f)(s) g^*_k(s) ds\big|$. By the Hölder inequality,

$$\begin{aligned}
&\|D^\gamma_s T^c_{\sigma_k}(f)\|^2_{L_2(\mathcal{N})}\\
&= \Big|\tau \int_{\mathbb{R}^d} D^\gamma_s T^c_{\sigma_k}(f)(s) g^*_k(s) ds\Big|^2\\
&= \Big|\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D^\gamma_s K_k(s, s-t) f_k(t)\, dt\, g^*_k(s)\, ds\Big|^2\\
&= \Big|\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|D^\gamma_s K_k(s, s-t)\|^{-\frac12}_{\mathcal{M}} D^\gamma_s K_k(s, s-t) f_k(t) \|D^\gamma_s K_k(s, s-t)\|^{\frac12}_{\mathcal{M}}\, g^*_k(s)\, dt\, ds\Big|^2\\
&\leq \tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|D^\gamma_s K_k(s, s-t)\|_{\mathcal{M}} |g_k(s)|^2\, dt\, ds \cdot \tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|D^\gamma_s K_k(s, s-t)\|_{\mathcal{M}} |f_k(t)|^2\, ds\, dt.
\end{aligned}$$

Then (3.8) and (3.9) ensure that

$$\|D^\gamma_s T^c_{\sigma_k}(f)\|^2_{L_2(\mathcal{N})} \lesssim 2^{2k|\gamma|_1} \cdot \|f_k\|^2_{L_2(\mathcal{N})}. \tag{3.10}$$

Taking $\gamma = 0$, the above calculation implies that

$$\|T^c_\sigma(f)\|_{L_2(\mathcal{N})} \leq \sum_{k\geq 0} \|T^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})} \lesssim \sum_{k\geq 0} \|f_k\|_{L_2(\mathcal{N})} \lesssim \|f\|_{B^0_{2,1}}, \tag{3.11}$$

which implies the boundedness of T^c_σ from $B^0_{2,1}(\mathbb{R}^d; L_2(\mathcal{M}))$ to $L_2(\mathcal{N})$.

On the other hand, if we take

$$a_0=\varphi_0,\quad a_j(\xi)=(1-\varphi_0(\xi))\frac{\xi_j}{|\xi|^2},$$

then we get

$$1=a_0(\xi)+\sum_{j=1}^d a_j(\xi)\xi_j,\quad \forall\,\xi\in\mathbb{R}^d.$$

This identity implies

$$1=(a_0(\xi)+\sum_{j=1}^d a_j(\xi)\xi_j)^l=\sum_{|\gamma|_1\le l}\sigma_\gamma(\xi)\xi^\gamma,\quad \forall\,l\in\mathbb{N}_0,\,\forall\,\xi\in\mathbb{R}^d,$$

where the $\sigma_\gamma(\xi)$'s are symbols in $S^{-|\gamma|_1}_{1,0}\subset S^{-|\gamma|_1}_{1,1}$. The above identity allows us to decompose the term $\varphi_j*T^c_{\sigma_k}(f)$ in the following way:

$$\varphi_j*T^c_{\sigma_k}(f)=\sum_{|\gamma|_1\le l}T^c_{\sigma_\gamma}(\varphi_j*D^\gamma_sT^c_{\sigma_k}(f))=\sum_{|\gamma|_1\le l}T^c_{\sigma^j_\gamma}(D^\gamma_sT^c_{\sigma_k}(f)),\tag{3.12}$$

where $\sigma^j_\gamma=\sigma_\gamma\widehat{\varphi}_j$. Note that the symbol $\sigma^j_\gamma\in S^{-|\gamma|_1}_{1,0}$ for any j, and if $|\gamma|_1<l$, $\sigma^j_\gamma\neq0$ if and only if $j=0$ and $j=1$. If $j\le k+1$, by the Plancherel formula and (3.10), we have

$$2^{j\alpha}\|\varphi_j*T^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})}\lesssim 2^{j\alpha}\|T^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})}\lesssim 2^{j\alpha}\|f_k\|_{L_2(\mathcal{N})}\lesssim 2^{k\alpha}\|f_k\|_{L_2(\mathcal{N})}.$$

If $j\ge k+2$, adapting the proof of (3.10) with σ^j_γ in place of σ_k, we deduce that

$$\|T^c_{\sigma^j_\gamma}(D^\gamma_sT^c_{\sigma_k}(f))\|_{L_2(\mathcal{N})}\le C_\gamma 2^{-j|\gamma|_1}\|D^\gamma_sT^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})}.\tag{3.13}$$

For any $|\gamma|_1<l$, by the previous observation, $\sigma^j_\gamma=0$. Therefore, estimates (3.10), (3.12) and (3.13) implies that

$$\begin{aligned}\|\varphi_j*T^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})}&=\|\sum_{|\gamma|_1=l}T^c_{\sigma^j_\gamma}(D^\gamma_sT^c_{\sigma_k}(f))\|_{L_2(\mathcal{N})}\\&\lesssim\sum_{|\gamma|_1=l}2^{-jl}\|D^\gamma_sT^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})}\\&\lesssim\sum_{|\gamma|_1=l}2^{(k-j)l}\|f_k\|_{L_2(\mathcal{N})}.\end{aligned}$$

Thus, if we take l to be the smallest integer larger than α, we have

$$2^{j\alpha}\|\varphi_j*T^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})}\lesssim 2^{(j-k)(\alpha-l)}2^{k\alpha}\|f_k\|_{L_2(\mathcal{N})}\le 2^{k\alpha}\|f_k\|_{L_2(\mathcal{N})}.$$

Combining the above estimate for $j\ge k+2$ and that for $j\le k+1$, we get

$$\sup_{j\in\mathbb{N}_0}2^{j\alpha}\|\varphi_j*T^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})}\lesssim 2^{k\alpha}\|f_k\|_{L_2(\mathcal{N})},$$

whence,

$$\|T^c_{\sigma_k}(f)\|_{B^\alpha_{2,\infty}}\lesssim 2^{k\alpha}\|f_k\|_{L_2(\mathcal{N})}.$$

Then by the triangle inequality, we have

$$\|T^c_\sigma(f)\|_{B^\alpha_{2,\infty}}\le\sum_{k\ge0}\|T^c_{\sigma_k}(f)\|_{B^\alpha_{2,\infty}}\lesssim\sum_{k\ge0}2^{k\alpha}\|f_k\|_{L_2(\mathcal{N})}\lesssim\|f\|_{B^\alpha_{2,1}},\tag{3.14}$$

which shows that T^c_σ is bounded from $B^\alpha_{2,1}(\mathbb{R}^d;L_2(\mathcal{M}))$ to $B^\alpha_{2,\infty}(\mathbb{R}^d;L_2(\mathcal{M}))$.

Applying (3.11), (3.14) and the real interpolation (3.1) with $p=2$, $q=2$ and $\alpha_0=0$, $\alpha_1=\alpha$, we obtain the following boundedness:

$$\|T^c_\sigma(f)\|_{B^\beta_{2,2}}\lesssim\|f\|_{B^\beta_{2,2}},\quad\forall\,\beta>0.$$

Finally, (3.7) together with the above inequality yields the desired assertion. □

3.2. **Mapping properties on Besov spaces.** Using a similar argument as in the proof of Proposition 3.4, we are able to obtain the regularity of pseudo-differential operators on general operator-valued Besov spaces $B^\alpha_{p,q}(\mathbb{R}^d; L_p(\mathcal{M}))$ with $1 \le p, q \le \infty$. Let us record it specifically below.

Theorem 3.5. *Let $1 \le p, q \le \infty$.*

i) *If $\sigma \in S^0_{1,\delta}$ for some $0 \le \delta \le 1$, then T^c_σ is bounded from $B^0_{p,1}(\mathbb{R}^d; L_p(\mathcal{M}))$ to $L_p(\mathcal{N})$, and bounded on $B^\alpha_{p,q}(\mathbb{R}^d; L_p(\mathcal{M}))$ for any $\alpha > 0$.*

ii) *If $\sigma \in S^0_{1,\delta}$ with $0 \le \delta < 1$, then T^c_σ is bounded on $B^\alpha_{p,q}(\mathbb{R}^d; L_p(\mathcal{M}))$ for any $\alpha \in \mathbb{R}$.*

iii) *The above assertions hold for T^r_σ as well.*

Proof. Firstly we note that the argument in (3.10) still works for all $1 \le p \le \infty$: For any $f \in H^\alpha_p(\mathbb{R}^d; L_p(\mathcal{M}))$ and $k \in \mathbb{N}_0$, there exists a norm one element $g_k \in L_q(\mathcal{N})$ with $\frac{1}{p} + \frac{1}{q} = 1$ such that $\|D^\gamma_s T^c_{\sigma_k}(f)\|_{L_p(\mathcal{N})} = \left|\tau \int_{\mathbb{R}^d} D^\gamma_s T^c_{\sigma_k}(f)(s) g^*_k(s) ds\right|$. Applying (3.8) and (3.9) again, we have

$$
\begin{aligned}
&\|D^\gamma_s T^c_{\sigma_k}(f)\|_{L_p(\mathcal{N})}\\
&= \Big|\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D^\gamma_s K_k(s, s-t) f_k(t)\, dt\, g^*_k(s)\, ds\Big|\\
&= \Big|\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|D^\gamma_s K_k(s, s-t)\|_{\mathcal{M}}^{\frac{-p+1}{p}} D^\gamma_s K_k(s, s-t) f_k(t) \|D^\gamma_s K_k(s, s-t)\|_{\mathcal{M}}^{\frac{p-1}{p}} g^*_k(s)\, dt\, ds\Big|\\
&\le \Big(\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|D^\gamma_s K_k(s, s-t)\|_{\mathcal{M}} |f_k(t)|^p\, dt\, ds\Big)^{\frac{1}{p}} \cdot \Big(\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|D^\gamma_s K_k(s, s-t)\|_{\mathcal{M}} |g_k(s)|^q\, dt\, ds\Big)^{\frac{1}{q}}\\
&\lesssim 2^{k|\gamma|_1} \cdot \|f_k\|_{L_p(\mathcal{N})}.
\end{aligned}
$$

Then we get the boundedness of T^c_σ from $B^0_{p,1}(\mathbb{R}^d; L_p(\mathcal{M}))$ to $L_p(\mathcal{N})$ as in (3.11). Furthermore, we can deduce the L_p-version of (3.14), i.e. the boundedness from $B^\alpha_{p,1}$ to $B^\alpha_{p,\infty}$ for $\alpha > 0$. Thus, for $\alpha > 0$, the boundedness of T^c_σ on $B^\alpha_{p,q}(\mathbb{R}^d; L_p(\mathcal{M}))$ is ensured by interpolation (3.1). The conclusion i) is therefore proved. If $\delta < 1$, by Proposition 2.8 and the lifting property of Besov spaces (see [1, Theorem 6.1]), i) yields ii) for general $\alpha \in \mathbb{R}$. Finally, the assertion for T^r_σ can be proved using the same method; we omit the details. □

4. The action of pseudo-differential operators on (sub)atoms

In order to study the boundedness of pseudo-differential operators on the Triebel-Lizorkin spaces, we will use the atomic decomposition stated in Theorem 1.3. In other words, we will focus on the images of the atoms under the action of pseudo-differential operators instead of those of general functions in the Triebel-Lizorkin spaces. Our idea initially comes from Triebel's book [53, Theorem 6.3.2], where the atomic decomposition is a key tool to treat the operators with symbols of forbidden type. However, due to the noncommutativity, the (sub)atoms we have obtained in our previous paper [56] (mentioned in Theorem 1.3) are L_2-atoms which do not have the pointwise estimates as the ones in [53]. Thus, it turns out that we need much subtler estimates regarding the images of these (sub)atoms in order to realise the required boundedness.

The first lemma in this section concerns the image of an $(\alpha, Q_{\mu,l})$-subatom under the action of pseudo-differential operators.

Lemma 4.1. *Let $\alpha \in \mathbb{R}$, $\sigma \in S^0_{1,\delta}$ and T^c_σ be the corresponding pseudo-differential operator. In addition, we assume that $K > \frac{d}{2}$. Then for any $(\alpha, Q_{\mu,l})$-subatom $a_{\mu,l}$, we have*

$$
\tau\Big(\int_{\mathbb{R}^d} (1 + 2^\mu |s - 2^{-\mu} l|)^{d+M} |D^\gamma T^c_\sigma a_{\mu,l}(s)|^2 ds\Big)^{\frac{1}{2}} \lesssim |Q_{\mu,l}|^{\frac{\alpha}{d} - \frac{|\gamma|_1}{d}}, \quad |\gamma|_1 < K - \frac{d}{2}, \tag{4.1}
$$

where $M \in \mathbb{R}$ such that $M < 2L + 2$ and the relevant constant depends on M, K, L, γ and d.

Proof. We split the integral on the left hand side of (4.1) into $\int_{4Q_{\mu,l}}$ and $\int_{(4Q_{\mu,l})^c}$. To estimate the term with $\int_{4Q_{\mu,l}}$, we begin with a technical modification of $a_{\mu,l}$. For every $a_{\mu,l}$, we define

$$
a = |Q_{\mu,l}|^{-\frac{\alpha}{d} + \frac{1}{2}} a_{\mu,l}(2^{-\mu}(\cdot + l)).
$$

It is easy to see that a is an $(\alpha, Q_{0,0})$-subatom. By translation, we may assume that $l=0$. Then by the Cauchy-Schwarz inequality (1.5), for any $s\in\mathbb{R}^d$, we have

$$\begin{aligned}|T^c_\sigma a_{\mu,l}(s)|^2 &= 2^{-2\mu d}|Q_{\mu,l}|^{2(\frac{\alpha}{d}-\frac12)}\Big|\int_{\mathbb{R}^d}\sigma(s,\xi)\widehat{a}(2^{-\mu}\xi)e^{2\pi \mathrm{i}s\cdot\xi}d\xi\Big|^2\\ &= |Q_{\mu,l}|^{2(\frac{\alpha}{d}-\frac12)}\Big|\int_{\mathbb{R}^d}\sigma(s,2^\mu\xi)\widehat{a}(\xi)e^{2\pi \mathrm{i}s\cdot 2^\mu\xi}d\xi\Big|^2\\ &\leq |Q_{\mu,l}|^{2(\frac{\alpha}{d}-\frac12)}\int_{\mathbb{R}^d}\|\sigma(s,2^\mu\xi)\|^2_{\mathcal{M}}(1+|\xi|^2)^{-K}d\xi\\ &\quad\int_{\mathbb{R}^d}(1+|\xi|^2)^K\|\sigma(s,2^\mu\xi)\|^{-2}_{\mathcal{M}}\,\widehat{a}^*(\xi)|\sigma(s,2^\mu\xi)|^2\widehat{a}(\xi)d\xi.\end{aligned}$$

Using the standard operator-valued inequality

$$x^*y^*yx\leq\|y\|^2x^*x, \tag{4.2}$$

we deduce from the Plancherel formula (1.6) that

$$\begin{aligned}|T^c_\sigma a_{\mu,l}(s)|^2 &\lesssim |Q_{\mu,l}|^{2(\frac{\alpha}{d}-\frac12)}\int_{\mathbb{R}^d}(1+|\xi|^2)^{-K}d\xi\cdot\int_{\mathbb{R}^d}(1+|\xi|^2)^K|\widehat{a}(\xi)|^2d\xi\\ &\lesssim |Q_{\mu,l}|^{2(\frac{\alpha}{d}-\frac12)}\int_{\mathbb{R}^d}|J^Ka(t)|^2dt,\end{aligned}$$

where J^K is the Bessel potential of order K. Combining the second assumption on $a_{\mu,l}$ in Definition 1.2 and the above estimate, we obtain

$$\begin{aligned}\tau\Big(\int_{4Q_{\mu,l}}|T^c_\sigma a_{\mu,l}(s)|^2(1+2^\mu|s|)^{d+M}ds\Big)^{\frac12} &\lesssim \tau\Big(\int_{4Q_{\mu,l}}|T^c_\sigma a_{\mu,l}(s)|^2ds\Big)^{\frac12}\\ &\lesssim |Q_{\mu,l}|^{\frac{\alpha}{d}}\tau\Big(\int_{\mathbb{R}^d}|J^Ka(t)|^2dt\Big)^{\frac12}\\ &\lesssim |Q_{\mu,l}|^{\frac{\alpha}{d}}\sum_{|\gamma|_1\leq K}\tau\Big(\int_{\mathbb{R}^d}|D^\gamma a(t)|^2dt\Big)^{\frac12}\lesssim |Q_{\mu,l}|^{\frac{\alpha}{d}}.\end{aligned}$$

If $s\in(4Q_{\mu,l})^c$, since $a_{\mu,l}$ has the moment cancellations of order less than or equal to L, we can subtract a Taylor polynomial of degree L from the kernel associated to T^c_σ,

$$\begin{aligned}|T^c_\sigma a_{\mu,l}(s)|^2 &= \Big|\int_{\mathbb{R}^d}K(s,s-t)a_{\mu,l}(t)dt\Big|^2\\ &= \Big|\int_{\mathbb{R}^d}[K(s,s-t)-K(s,s)]a_{\mu,l}(t)dt\Big|^2\\ &= \Big|\int_{\mathbb{R}^d}\Big[\sum_{|\beta|_1=L+1}\frac{L+1}{\beta!}t^\beta\int_0^1(1-\theta)^LD^\beta K(s,s-\theta t)d\theta\Big]a_{\mu,l}(t)dt\Big|^2.\end{aligned}$$

Applying the Cauchy-Schwarz inequality (1.5) and then the inequalities (2.7) and (4.2), we get

$$\begin{aligned}|T^c_\sigma a_{\mu,l}(s)|^2 &\lesssim \sum_{|\beta|_1=L+1}\int_{2Q_{\mu,l}}\Big\|\int_0^1(1-\theta)^LD^\beta K(s,s-\theta t)d\theta\Big\|^2_{\mathcal{M}}|t|^{2L+2}dt\\ &\quad\cdot\int_{\mathbb{R}^d}\Big\|\int_0^1(1-\theta)^LD^\beta K(s,s-\theta t)d\theta\Big\|^{-2}_{\mathcal{M}}\Big|\int_0^1(1-\theta)^LD^\beta K(s,s-\theta t)d\theta\, a_{\mu,l}(t)\Big|^2dt\\ &\leq \sum_{|\beta|_1=L+1}\int_{2Q_{\mu,l}}\sup_{0\leq\theta\leq1}\|D^\beta K(s,s-\theta t)\|^2_{\mathcal{M}}\,|t|^{2L+2}dt\cdot\int_{\mathbb{R}^d}|a_{\mu,l}(t)|^2dt \qquad (4.3)\\ &\lesssim |s|^{-2d-2L-2}\int_{2Q_{\mu,l}}|t|^{2L+2}dt\cdot\int_{\mathbb{R}^d}|a_{\mu,l}(t)|^2dt\\ &\lesssim 2^{-\mu(2L+2+d)}|s|^{-2d-2L-2}\int_{\mathbb{R}^d}|a_{\mu,l}(t)|^2dt.\end{aligned}$$

This estimate implies

$$\begin{aligned}
&\tau\Big(\int_{(4Q_{\mu,l})^c} |T_\sigma^c a_{\mu,l}(s)|^2(1+2^\mu|s|)^{d+M}ds\Big)^{\frac12}\\
&\lesssim 2^{-\mu(L+1-\frac{M}{2})}\Big(\int_{(4Q_{\mu,l})^c}|s|^{-d-2L-2+M}ds\Big)^{\frac12}\cdot\tau\Big(\int_{\mathbb{R}^d}|a_{\mu,l}(t)|^2dt\Big)^{\frac12}\\
&\lesssim 2^{-\mu(L+1-\frac{M}{2})}2^{\mu(L+1-\frac{M}{2})}|Q_{\mu,l}|^{\frac{\alpha}{d}}=|Q_{\mu,l}|^{\frac{\alpha}{d}}.
\end{aligned}$$

If we take $M=-d$ in the above inequality, we have $T_\sigma^c a_{\mu,l}\in L_1\big(\mathcal{M};L_2^c(\mathbb{R}^d)\big)$. By approximation, we can assume that $\sigma(s,\xi)$ has compact ξ-support, so that

$$T_\sigma^c a_{\mu,l}(s)=\int_{\mathbb{R}^d}\sigma(s,\xi)\widehat{a_{\mu,l}}(\xi)e^{2\pi \mathrm{i} s\cdot\xi}d\xi$$

is uniformly convergent. Moreover, one can differentiate the integrand and obtain always uniformly convergent integrals. Then, for any $|\gamma|_1<K-\frac{d}{2}$, we have

$$\begin{aligned}
&\tau\Big(\int_{4Q_{\mu,l}}|D^\gamma T_\sigma^c a_{\mu,l}(s)|^2(1+2^\mu|s|)^{d+M}ds\Big)^{\frac12}\\
&\lesssim |Q_{\mu,l}|^{\frac{\alpha}{d}}\tau\Big(\int_{\mathbb{R}^d}|J^K a(t)|^2dt\Big)^{\frac12}\int_{\mathbb{R}^d}(1+|\xi|)^{2|\gamma|_1-2K}d\xi \qquad (4.4)\\
&\lesssim |Q_{\mu,l}|^{\frac{\alpha}{d}}\sum_{|\gamma|_1\le K}\tau\Big(\int_{\mathbb{R}^d}|D^\gamma a(t)|^2dt\Big)^{\frac12}\lesssim|Q_{\mu,l}|^{\frac{\alpha}{d}}.
\end{aligned}$$

By a similar argument to that of (4.3), we have, for any $\gamma\in\mathbb{N}_0^d$ and $s\in(4Q_{\mu,l})^c$,

$$|D^\gamma T_\sigma^c a_{\mu,l}(s)|^2\lesssim 2^{-\mu(2L+2+d)}|s|^{-2d-2L-2-2|\gamma|_1}\int_{\mathbb{R}^d}|a_{\mu,l}(t)|^2dt. \qquad (4.5)$$

Therefore, we deduce that

$$\begin{aligned}
&\tau\Big(\int_{(4Q_{\mu,l})^c}|D^\gamma T_\sigma^c a_{\mu,l}(s)|^2(1+2^\mu|s|)^{d+M}ds\Big)^{\frac12}\\
&\lesssim 2^{-\mu(L+1-\frac{M}{2})}\Big(\int_{(4Q_{\mu,l})^c}|s|^{-d-2L-2+M-2|\gamma|_1}ds\Big)^{\frac12}\cdot\tau\Big(\int_{\mathbb{R}^d}|a_{\mu,l}(t)|^2dt\Big)^{\frac12}\\
&\lesssim 2^{-\mu(L+1-\frac{M}{2})}2^{\mu(L+1-\frac{M}{2}+|\gamma|_1)}|Q_{\mu,l}|^{\frac{\alpha}{d}}=|Q_{\mu,l}|^{\frac{\alpha}{d}-\frac{|\gamma|_1}{d}}.
\end{aligned}$$

Combining the estimates above, we get (4.1). □

On the other hand, we also have the following lemma concerning the image of $(\alpha,1)$-atoms under the action of pseudo-differential operators.

Lemma 4.2. *Let $\alpha\in\mathbb{R}$, $\sigma\in S^0_{1,\delta}$. Let $K>\frac{d}{2}$ and b be an $(\alpha,1)$-atom based on the cube $Q_{0,m}$. Then for any $M\in\mathbb{R}$, we have*

$$\tau\Big(\int_{\mathbb{R}^d}(1+|s-m|)^{d+M}|D^\gamma T_\sigma^c b(s)|^2ds\Big)^{\frac12}\lesssim1,\quad |\gamma|_1<K-\frac{d}{2}, \qquad (4.6)$$

where the relevant constant depends on M, K, γ and d.

Proof. The proof of this lemma is similar to that of the previous one. The only difference is that for an $(\alpha,1)$-atom, we do not necessarily have the moment cancellation; thus, we have to use the extra decay of the kernel proved in Lemma 2.6 for $|t|>1$.

If $s\in 4Q_{0,m}$, we follow the estimate for subatoms in the previous lemma. Applying the size estimate of b, we get

$$\tau\Big(\int_{4Q_{0,m}}(1+|s-m|)^{d+M}|T_\sigma^c b(s)|^2ds\Big)^{\frac12}\lesssim\tau\Big(\int_{\mathbb{R}^d}|J^K b(t)|^2dt\Big)^{\frac12}\lesssim1.$$

If $s \in (4Q_{0,m})^c$ and $t \in 2Q_{0,m}$, we have $|s-t| \geq 1$. Then (2.8) gives

$$
\begin{aligned}
|T^c_\sigma b(s)|^2 &= \Big| \int_{\mathbb{R}^d} K(s, s-t) b(t) dt \Big|^2 \\
&\leq \int_{2Q_{0,m}} \|K(s,s-t)\|^2_{\mathcal{M}} dt \int_{2Q_{0,m}} |b(t)|^2 dt \\
&\lesssim |s-m|^{-2N} \int_{2Q_{0,m}} |b(t)|^2 dt,
\end{aligned}
$$

where the positive integer N can be arbitrarily large. Thus

$$
\begin{aligned}
&\tau\Big(\int_{(4Q_{0,m})^c} (1+|s-m|)^{d+M} |T^c_\sigma b(s)|^2 ds \Big)^{\frac{1}{2}} \\
&\lesssim \Big(\int_{(4Q_{0,m})^c} |s-m|^{d+M-2N} ds \Big)^{\frac{1}{2}} \tau\Big(\int_{2Q_{0,m}} |b(t)|^2 dt \Big)^{\frac{1}{2}} \lesssim 1.
\end{aligned}
$$

Then, the estimates obtained above imply that

$$
\tau\Big(\int_{\mathbb{R}^d} (1+|s-m|)^{d+M} |T^c_\sigma b(s)|^2 ds \Big)^{\frac{1}{2}} \lesssim 1.
$$

Similarly, we treat $D^\gamma T^c_\sigma b(s)$ as

$$
\tau\Big(\int_{\mathbb{R}^d} (1+|s-m|)^{d+M} |D^\gamma T^c_\sigma b(s)|^2 ds \Big)^{\frac{1}{2}} \lesssim 1, \quad |\gamma|_1 < K - \frac{d}{2}.
$$

Therefore, (4.6) is proved. □

The following lemma shows that, if the symbol σ satisfies some support condition, we can even estimate the $F_1^{\alpha,c}$-norm of the image of $(\alpha, Q_{\mu,l})$-subatoms under T^c_σ. Recall that for any cube $Q \subset \mathbb{R}^d$ and any $s \in \mathbb{R}^d$, $s+Q$ denotes the cube $\{t \in \mathbb{R}^d : t - s \in Q\}$.

Lemma 4.3. *Let $\sigma \in S^0_{1,\delta}$ and $\alpha \in \mathbb{R}$. Assume that $K \in \mathbb{N}$ satisfy $K > \frac{d}{2}$ and $K > \alpha + d + 1$. If the s-support of σ is in $(2^{-\mu} l + 4Q_{0,0})^c$, then for any $(\alpha, Q_{\mu,l})$-subatom $a_{\mu,l}$, we have*

$$
\|T^c_\sigma a_{\mu,l}\|_{F_1^{\alpha,c}} \lesssim 2^{-\mu(\frac{d}{2}+\iota)},
$$

where ι is a positive real number.

Proof. Without loss of generality, we still assume $l = 0$. We need to use the characterization of $F_1^{\alpha,c}$-norm by the following convolution kernels. Let κ be a radial, real and infinitely differentiable function on $\mathbb{R}^d$ supported in $Q_{0,0}$, and assume that $\widehat{\kappa}(0) > 0$. We take $\Phi = |\cdot|^N \widehat{\kappa}$ with $N \in \mathbb{N}_0$ such that $\alpha + \frac{d}{2} < N < K - \frac{d}{2}$, and another test function $\Phi_0 \in \mathcal{S}$ with $\operatorname{supp} \Phi_0 \subset Q_{0,0}$. Let Φ_ε be the inverse Fourier transform of $\Phi(\varepsilon\xi)$. To simplify the notation, we denote $T^c_\sigma a_{\mu,l}$ by $\eta_{\mu,l}$. Then Theorem 4.2 in [56] yields

$$
\|\eta_{\mu,l}\|_{F_1^{\alpha,c}} \approx \|\Phi_0 * \eta_{\mu,l}\|_1 + \Big\| \Big(\int_0^1 \varepsilon^{-2\alpha} |\Phi_\varepsilon * \eta_{\mu,l}|^2 \frac{d\varepsilon}{\varepsilon} \Big)^{\frac{1}{2}} \Big\|_1.
$$

We notice that $\int_{\mathbb{R}^d} \Phi_\varepsilon(t) t^\gamma dt = 0$ for any $|\gamma|_1 \leq N-1$, it follows that

$$
\begin{aligned}
\Phi_\varepsilon * \eta_{\mu,l}(s) &= \int_{\mathbb{R}^d} \Phi_\varepsilon(t) [\eta_{\mu,l}(s-t) - \eta_{\mu,l}(s)] dt \\
&= \int_{\mathbb{R}^d} \Phi_\varepsilon(t) \sum_{|\gamma|_1 = N} \frac{N}{\gamma!} (-t)^\gamma \int_0^1 (1-\theta)^{N-1} D^\gamma \eta_{\mu,l}(s - \theta t) d\theta \, dt.
\end{aligned} \tag{4.7}
$$

Applying the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\Big| \int_{\mathbb{R}^d} \Phi_\varepsilon(t) \sum_{|\gamma|_1 = N} \frac{N}{\gamma!} (-t)^\gamma \int_0^1 (1-\theta)^{N-1} D^\gamma \eta_{\mu,l}(s-\theta t) d\theta \, dt \Big|^2 \\
&\lesssim \sum_{|\gamma|_1 = N} \int_{\mathbb{R}^d} \int_0^1 (1-\theta)^{2(N-1)} |D^\gamma \eta_{\mu,l}(s-\theta t)|^2 d\theta (1+|t|)^{-d-1} dt \\
&\quad \cdot \int_{\mathbb{R}^d} |\Phi_\varepsilon(t)|^2 |t|^{2N} (1+|t|)^{d+1} dt.
\end{aligned} \tag{4.8}
$$

By (4.5), if $s-\theta t\in(4Q_{0,0})^c$, we have

$$|D^\gamma\eta_{\mu,l}(s-\theta t)|^2\lesssim 2^{-\mu(2L+2+d)}|s-\theta t|^{-2d-2L-2-2|\gamma|_1}\int_{\mathbb{R}^d}|a_{\mu,l}(r)|^2dr.$$

Therefore, using the Cauchy-Schwarz inequality again, we have

$$\begin{aligned}
&\Big\|\Big(\int_0^1\varepsilon^{-2\alpha}|\Phi_\varepsilon*\eta_{\mu,l}|^2\frac{d\varepsilon}{\varepsilon}\Big)^{\frac12}\Big\|_1\\
&\lesssim 2^{-\mu(L+1+\frac d2)}\Big(\int_0^1\varepsilon^{2N-d-2\alpha}\frac{d\varepsilon}{\varepsilon}\Big)^{\frac12}\int_{(2Q_{0,0})^c}|s'|^{-d-L-N-1}ds'\int_{\mathbb{R}^d}(1+|t|)^{-d-1}dt\\
&\quad\cdot\int_{\mathbb{R}^d}|\Phi(t')|^2|t'|^{2N}(1+|t'|)^{d+1}dt'\cdot\tau\Big(\int_{\mathbb{R}^d}|a_{\mu,l}(r)|^2dr\Big)^{\frac12}\\
&\lesssim 2^{-\mu(L+1+\frac d2)}\tau\Big(\int_{\mathbb{R}^d}|a_{\mu,l}(t)|^2dt\Big)^{\frac12}\\
&\lesssim 2^{-\mu(L+1+\frac d2+\alpha)}.
\end{aligned}\tag{4.9}$$

It remains to estimate the L_1-norm of $\Phi_0*\eta_{\mu,l}$, where Φ_0 does not have moment cancellation. Since $\operatorname{supp}\eta_{\mu,l}\subset(4Q_{0,0})^c$ and by the support assumption of Φ_0, we have $\operatorname{supp}\Phi_0*\eta_{\mu,l}\subset\{s\in\mathbb{R}^d:|s|\geq\frac12\}$. By Lemma 4.1 and the fact that $|\Phi_0(s)|\lesssim(1+|s|)^{-d-R}$ for any $R\in\mathbb{N}$, we have

$$\begin{aligned}
&|\Phi_0*\eta_{\mu,l}(s)|^2=\Big|\int_{\mathbb{R}^d}\Phi_0(s-t)\eta_{\mu,l}(t)dt\Big|^2\\
&\leq\Big|\int_{|t|\geq\max\{\frac{|s|}{2},1\}}\Phi_0(s-t)\eta_{\mu,l}(t)dt\Big|^2+\Big|\int_{1\leq|t|<\frac{|s|}{2}}\Phi_0(s-t)\eta_{\mu,l}(t)dt\Big|^2\\
&\leq\int_{|t|\geq\max\{\frac{|s|}{2},1\}}(1+2^\mu|t|)^{-2d-2R}|\Phi_0(s-t)|^2dt\cdot\int_{\mathbb{R}^d}(1+2^\mu|t|)^{2d+2R}|\eta_{\mu,l}(t)|^2dt\\
&\quad+\int_{1\leq|t|<\frac{|s|}{2}}(1+2^\mu|t|)^{-2d-2R}|\Phi_0(s-t)|^2dt\cdot\int_{\mathbb{R}^d}(1+2^\mu|t|)^{2d+2R}|\eta_{\mu,l}(t)|^2dt\\
&\lesssim\int_{\mathbb{R}^d}|\Phi_0(t)|^2dt\,(1+2^\mu|s|)^{-2d-2R}\int_{\mathbb{R}^d}(1+2^\mu|t|)^{2d+2R}|\eta_{\mu,l}(t)|^2dt\\
&\quad+\int_{|t|\geq1}(1+2^\mu|t|)^{-2d-2R}dt\,(1+|s|)^{-2d-2R}\int_{\mathbb{R}^d}(1+2^\mu|t|)^{2d+2R}|\eta_{\mu,l}(t)|^2dt\\
&\lesssim\Big((1+2^\mu|s|)^{-2d-2R}+2^{-2\mu(d+R)}(1+|s|)^{-2d-2R}\Big)\int_{\mathbb{R}^d}(1+2^\mu|t|)^{2d+2R}|\eta_{\mu,l}(t)|^2dt.
\end{aligned}$$

Then we use (4.1) to get, for any $R\in\mathbb{N}$,

$$\begin{aligned}
\|\Phi_0*\eta_{\mu,l}\|_1&\lesssim\Big(\int_{|s|\geq\frac12}(1+2^\mu|s|)^{-d-R}ds+2^{-\mu(d+R)}\int_{|s|\geq\frac12}(1+|s|)^{-d-R}ds\Big)\\
&\quad\cdot\tau\Big(\int_{\mathbb{R}^d}(1+2^\mu|t|)^{2d+2R}|\eta_{\mu,l}(t)|^2dt\Big)^{\frac12}\\
&\lesssim 2^{-\mu(d+R+\alpha)}.
\end{aligned}$$

Combining the estimates above, we see that, there exists $\iota>0$ such that

$$\|T_\sigma^ca_{\mu,l}\|_{F_1^{\alpha,c}}=\|\eta_{\mu,l}\|_{F_1^{\alpha,c}}\lesssim 2^{-\mu(\frac d2+\iota)},$$

which completes the proof. □

Since every $(\alpha,Q_{k,m})$-atom is a linear combination of subatoms, the above lemma helps us to estimate the image of $(\alpha,Q_{k,m})$-atoms under T_σ^c.

Corollary 4.4. *Let $\sigma\in S^0_{1,\delta}$ and $\alpha\in\mathbb{R}$. Assume that $K\in\mathbb{N}$ satisfy $K>\frac d2$ and $K>\alpha+d+1$. If the s-support of σ is in $(2^{-k}m+6Q_{0,0})^c$, then for any $(\alpha,Q_{k,m})$-atom g,we have*

$$\|T_\sigma^cg\|_{F_1^{\alpha,c}}\lesssim1.$$

Proof. Every $(\alpha, Q_{k,m})$-atom g admits the form

$$g = \sum_{(\mu,l)\le(k,m)} d_{\mu,l} a_{\mu,l} \quad \text{with} \quad \sum_{(\mu,l)\le(k,m)} |d_{\mu,l}|^2 \le |Q_{k,m}|^{-1} = 2^{kd}.$$

By the support assumption of σ, $\sigma(s,\xi) = 0$ if $s \in 2^{-\mu}l + 4Q_{0,0} \subset 2^{-k}m + 6Q_{0,0}$. Then, we can apply the previous lemma to every $a_{\mu,l}$ with $(\mu,l) \le (k,m)$. The result is

$$\|T^c_\sigma a_{\mu,l}\|_{F_1^{\alpha,c}} \lesssim 2^{-\mu(\frac{d}{2}+\iota)} \quad \text{with } \iota > 0.$$

Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\|T^c_\sigma g\|_{F_1^{\alpha,c}} &\le \sum_{(\mu,l)\le(k,m)} |d_{\mu,l}| \cdot \|T^c_\sigma a_{\mu,l}\|_{F_1^{\alpha,c}} \\
&\le \sum_{(\mu,l)\le(k,m)} |d_{\mu,l}| \cdot 2^{-\mu(\frac{d}{2}+\iota)} \\
&\lesssim \Big(\sum_{(\mu,l)\le(k,m)} |d_{\mu,l}|^2\Big)^{\frac12} \Big(\sum_{(\mu,l)\le(k,m)} 2^{-\mu(d+2\iota)}\Big)^{\frac12} \qquad (4.10)\\
&\le \Big(\sum_{(\mu,l)\le(k,m)} |d_{\mu,l}|^2\Big)^{\frac12} \Big(\sum_{\mu\ge k} \frac{|2Q_{k,m}|}{|Q_{\mu,l}|} \cdot 2^{-\mu(d+2\iota)}\Big)^{\frac12} \\
&\lesssim |Q_{k,m}|^{-\frac12} \cdot 2^{-\frac{kd}{2}} = 1.
\end{aligned}$$

Thus, the assertion is proved. □

Likewise, we can estimate the image of $(\alpha,1)$-atoms under the pseudo-differential operator T^c_σ.

Lemma 4.5. *Let $\sigma \in S^0_{1,\delta}$ and $\alpha \in \mathbb{R}$. Assume that $K \in \mathbb{N}$ satisfy $K > \frac{d}{2}$ and $K > \alpha + d + 1$. If the s-support of σ is in $(k + 4Q_{0,0})^c$ for some $k \in \mathbb{Z}^d$, then for any $(\alpha,1)$-atom b such that* $\operatorname{supp} b \subset 2Q_{0,k}$, *we have*

$$\|T^c_\sigma b\|_{F_1^{\alpha,c}} \lesssim 1.$$

Proof. The proof of this lemma is very similar to that of Lemma 4.3; it suffices to apply (the proof of) Lemma 4.2 instead of Lemma 4.1. □

Corollary 4.6. *Let $\sigma \in S^0_{1,\delta}$ and $\alpha \in \mathbb{R}$. Given $K \in \mathbb{N}$ such that $K > \frac{d}{2}$ and $K > \alpha + d + 1$, then for any $(\alpha,1)$-atom b, we have*

$$\|T^c_\sigma b\|_{F_1^{\alpha,c}} \lesssim 1.$$

Proof. Let $(\mathcal{X}_j)_{j\in\mathbb{Z}^d}$ be the smooth resolution of the unit in (1.4). We decompose $T^c_\sigma b$ as

$$T^c_\sigma b = \sum_{j\in\mathbb{Z}^d} \mathcal{X}_j T^c_\sigma b = \sum_{j\in\mathbb{Z}^d} T^c_{\sigma_j} b,$$

where all $\sigma_j = \mathcal{X}_j(s)\sigma(s,\xi)$ belong to $S^0_{1,\delta}$ uniformly in j. Suppose that b is supported in $2Q_{0,k}$ with $k \in \mathbb{Z}^d$. We split the above summation into two parts:

$$T^c_\sigma b = \sum_{\substack{j\in k+6Q_{0,0}\\ j\in\mathbb{Z}^d}} \mathcal{X}_j T^c_\sigma b + \sum_{\substack{j\notin k+6Q_{0,0}\\ j\in\mathbb{Z}^d}} \mathcal{X}_j T^c_\sigma b. \qquad (4.11)$$

Applying Lemma 4.2 with $M = -d$ to the symbol $\mathcal{X}_j(s)\sigma(s,\xi)$, we get, for any $j \in \mathbb{Z}^d$,

$$\tau\Big(\int_{j+2Q_{0,0}} |D^\gamma(\mathcal{X}_j\, T^c_\sigma b(s))|^2 ds\Big)^{\frac12} \lesssim 1, \quad \forall\, |\gamma|_1 \le [\alpha] + 1.$$

Thus, $\mathcal{X}_j\, T^c_\sigma b$ is a bounded multiple of an $(\alpha,1)$-atom. So the first term on the right hand side of (4.11) is a finite sum of $(\alpha,1)$-atoms, and thus has bounded $F_1^{\alpha,c}$-norm. Now we deal with the second term. Note that the s-support of the symbol $\sum_{j\notin k+6Q_{0,0}} \mathcal{X}_j(s)\sigma(s,\xi)$ is in $(k + 4Q_{0,0})^c$. Then, it suffices to apply Lemma 4.5 to this symbol, so that

$$\Big\| \sum_{\substack{j\notin k+6Q_{0,0}\\ j\in\mathbb{Z}^d}} \mathcal{X}_j T^c_\sigma b \Big\|_{F_1^{\alpha,c}} \lesssim 1.$$

The proof is complete. □

5. Regular symbols on Triebel-Lizorkin spaces

In this section, we study the continuity of the pseudo-differential operators with regular symbols in $S^0_{1,\delta}$ $(0 \leq \delta < 1)$ on Triebel-Lizorkin spaces. We use the atomic decompositions introduced in section 1 and the local mapping properties of pseudo-differential operators in section 4 to deduce the $F^{\alpha,c}_p$-boundedness. Different from the pseudo-differential operators with the forbidden symbols in $S^0_{1,1}$, which will be treated in the next section, our proof stays at the level of atoms; in other words, we do not need the subtler decomposition that every (α, Q)-atom can be written as a linear combination of subatoms.

Theorem 5.1. *Let $0 \leq \delta < 1$, $\sigma \in S^0_{1,\delta}$ and $\alpha \in \mathbb{R}$. Then T^c_σ is bounded on $F^{\alpha,c}_p(\mathbb{R}^d, \mathcal{M})$ for any $1 \leq p \leq \infty$.*

In order to fully understand the image of an (α, Q)-atom under the action of a pseudo-differential operator, we need to study its $L_1(\mathcal{M}; L^c_2(\mathbb{R}^d))$-boundedness, which relies on the L_2-boundedness of T^c_σ given in Corollary 3.3.

Lemma 5.2. *Let $\sigma \in S^0_{1,\delta}$ with $0 \leq \delta < 1$. Then T^c_σ is bounded on $L_1\big(\mathcal{M}; L^c_2(\mathbb{R}^d)\big)$.*

Proof. Since $0 \leq \delta < 1$, Proposition 2.9 tells us that the adjoint $(T^c_\sigma)^*$ of T^c_σ is still in the class $S^0_{1,\delta}$. Thus, by anti-linear duality (see [23]), it is enough to prove the boundedness of $(T^c_\sigma)^*$ on $L_\infty\big(\mathcal{M}; L^c_2(\mathbb{R}^d)\big)$. Indeed, there exists $u \in L_2(\mathcal{M})$ with norm one such that

$$\Big\| \Big(\int_{\mathbb{R}^d} |(T^c_\sigma)^*(f)(s)|^2 ds \Big)^{\frac12} \Big\|_{\mathcal{M}} = \Big(\int_{\mathbb{R}^d} \langle |(T^c_\sigma)^*(f)(s)|^2 u, u \rangle_{L_2(\mathcal{M})} ds \Big)^{\frac12}$$
$$= \Big(\int_{\mathbb{R}^d} \|(T^c_\sigma)^*(fu)(s)\|^2_{L_2(\mathcal{M})} ds \Big)^{\frac12}.$$

Then, applying Corollary 3.3 to $(T^c_\sigma)^*$, we get

$$\Big(\int_{\mathbb{R}^d} \|(T^c_\sigma)^*(fu)(s)\|^2_{L_2(\mathcal{M})} ds \Big)^{\frac12} \lesssim \Big(\int_{\mathbb{R}^d} \|f(s)u\|^2_{L_2(\mathcal{M})} ds \Big)^{\frac12} \leq \Big\| \Big(\int_{\mathbb{R}^d} |f(s)|^2 ds \Big)^{\frac12} \Big\|_{\mathcal{M}}.$$

Thus, we conclude the boundedness of T^c_σ on $L_1(\mathcal{M}; L^c_2(\mathbb{R}^d))$. □

Now we are ready to prove the main theorem in this section.

Proof of Theorem 5.1. Step 1. We begin with the special case $p = 1$ and $\alpha = 0$. Since $F^{0,c}_1(\mathbb{R}^d, \mathcal{M}) = \mathrm{h}^c_1(\mathbb{R}^d, \mathcal{M})$ with equivalent norms, the assertion is equivalent to saying that when $\sigma \in S^0_{1,\delta}$ with $0 \leq \delta < 1$, T^c_σ is bounded on $\mathrm{h}^c_1(\mathbb{R}^d, \mathcal{M})$. By the atomic decomposition stated in Theorem 1.3, it suffices to prove that, for any atom b based on a cube with side length 1 and any atom g based on a cube with side length less than 1, we have

$$\|T^c_\sigma b\|_{\mathrm{h}^c_1} \lesssim 1 \quad \text{and} \quad \|T^c_\sigma g\|_{\mathrm{h}^c_1} \lesssim 1.$$

Corollary 4.6 tells us that

$$\|T^c_\sigma b\|_{\mathrm{h}^c_1} \lesssim 1.$$

Thus, it remains to consider the atom g based on cube Q with $|Q| < 1$. Without loss of generality, we may assume that Q is centered at the origin. Let $(\mathcal{X}_j)_{j \in \mathbb{Z}^d}$ be the resolution of the unit defined in (1.4) and $\mathcal{X}^Q_j = \mathcal{X}_j(l(Q)^{-1}\cdot)$ for $j \in \mathbb{Z}^d$. Then, we have $\operatorname{supp} \mathcal{X}^Q_j \subset l(Q)j + 2Q$. Now, set $h_1 = \sum_{j \in 6Q_{0,0}} \mathcal{X}^Q_j$ and $h_2 = \sum_{j \notin 6Q_{0,0}} \mathcal{X}^Q_j$. By the support assumption of $\mathcal{X}^Q_j$, it is obvious that $\operatorname{supp} h_1 \subset 8Q$, $\operatorname{supp} h_2 \subset (4Q)^c$. Moreover,

$$h_1(s) + h_2(s) = 1 \quad , \forall\, s \in \mathbb{R}^d.$$

Now we decompose σ into two parts:

$$\sigma(s,\xi) = h_1(s)\sigma(s,\xi) + h_2(s)\sigma(s,\xi) \stackrel{\text{def}}{=} \sigma^1(s,\xi) + \sigma^2(s,\xi).$$

Note that σ^1 and σ^2 are still in the class $S^0_{1,\delta}$ and

$$T^c_\sigma g = T^c_{\sigma^1} g + T^c_{\sigma^2} g.$$

Firstly, we deal with the symbol σ^1 which has compact s-support. We consider the adjoint operator $(T^c_\sigma)^*$ of T^c_σ. Since $\delta<1$, by Proposition 2.9, there exists $\widetilde{\sigma}\in S^0_{1,\delta}$ such that

$$(T^c_\sigma)^* = T^c_{\widetilde{\sigma}}.$$

If we take $\zeta_j(s)=\mathcal{X}^Q_j(s)\widetilde{\sigma}(s,0)^*$ for $j\in 6Q_{0,0}$, then ζ_j is an $\mathcal{M}$-valued infinitely differentiable function with all derivatives belonging to $L_\infty(\mathcal{N})$. Denote by $m^c_{\zeta_j}$ the pointwise multiplication $g\mapsto\zeta_j g$. Then, we have

$$\operatorname{supp} m^c_{\zeta_j} g\subset l(Q)j+2Q.$$

and

$$\tau\Big(\int_{\mathbb{R}^d}|m^c_{\zeta_j}g(s)|^2ds\Big)^{\frac12}\lesssim|Q|^{-\frac12}. \tag{5.1}$$

This indicates that, except for the vanishing mean property, each $m^c_{\zeta_j}g$ coincides with a bounded multiple of an h^c_1-atom defined in Definition 1.1. Now let us set $\sigma^1_j(s,\xi)=\mathcal{X}^Q_j(s)\sigma(s,\xi)$ for $j\in 6Q_{0,0}$ and set $T^c_j=T^c_{\sigma^1_j}-m^c_{\zeta_j}$. It is clear that $\operatorname{supp}T^c_jg\subset l(Q)j+2Q$. Since $(m^c_{\zeta_j})^*=m^c_{\zeta_j{}^*}$ and $(T^c_{\sigma^1_j})^*x=\widetilde{\sigma^1_j}(s,0)x=\zeta_j{}^*x$ for every $x\in\mathcal{M}$, we have

$$\tau\Big(\int_{l(Q)j+2Q}T^c_jg(s)ds\cdot x\Big)=\langle T^c_jg,x\rangle=\langle g,(T^c_j)^*x\rangle=\langle g,(T^c_{\sigma^1_j}-m^c_{\zeta_j})^*x\rangle=0.$$

Hence, T^c_jg has vanishing mean. Moreover, applying Lemma 5.2 and (5.1), we get

$$\begin{aligned}\tau\Big(\int_{l(Q)j+2Q}|T^c_jg(s)|^2ds\Big)^{\frac12}&\le\tau\Big(\int_{l(Q)j+2Q}|T^c_{\sigma^1_j}g(s)|^2ds\Big)^{\frac12}+\tau\Big(\int_{l(Q)j+2Q}|m^c_{\zeta_j}g(s)|^2ds\Big)^{\frac12}\\&\lesssim\tau\Big(\int_{2Q}|g(s)|^2ds\Big)^{\frac12}+|Q|^{-\frac12}\lesssim|Q|^{-\frac12}.\end{aligned}$$

Combining the above estimates, we see that T^c_j maps h^c_1-atoms to h^c_1-atoms. Thus, T^c_j is bounded on $\mathrm{h}^c_1(\mathbb{R}^d,\mathcal{M})$, and so are $T^c_{\sigma^1_j}$ and $T^c_{\sigma^1}$.

Step 2. Now let us consider $T^c_{\sigma^2}$. By Theorem 1.3, we may assume that g has moment cancellations up to order $L>\frac d2-1$. Note that $\operatorname{supp}T^c_{\sigma^2}g\subset(4Q)^c$. And if $s\in(4Q)^c$, following the argument in (4.3) with g in place of $a_{\mu,l}$, we get

$$|T^c_{\sigma^2}g(s)|^2\lesssim l(Q)^{2L+2+d}|s|^{-2d-2L-2}\int_{2Q}|g(t)|^2dt.$$

Then for $M<2L+2$,

$$\begin{aligned}&\tau\Big(\int_{(4Q)^c}|T^c_{\sigma^2}g(s)|^2(1+l(Q)^{-1}|s|)^{d+M}ds\Big)^{\frac12}\\&\lesssim l(Q)^{L+1-\frac M2}\Big(\int_{(4Q)^c}|s|^{-d-2L-2+M}ds\Big)^{\frac12}\cdot\tau\Big(\int_{2Q}|g(t)|^2dt\Big)^{\frac12}\\&\lesssim l(Q)^{L+1-\frac M2}l(Q)^{-L-1+\frac M2}|Q|^{-\frac12}=|Q|^{-\frac12}.\end{aligned}\tag{5.2}$$

Moreover, we claim that $T^c_{\sigma^2}g$ can be decomposed as follows:

$$T^c_{\sigma^2}g=\sum_{m\in\mathbb{Z}^d}\nu_mH_m,$$

where $\sum_m|\nu_m|\lesssim1$ and the H_m's are h^c_1-atoms. Then, by (1.8), we will get $\|T^c_{\sigma^2}g\|_{\mathrm{h}^c_1}\lesssim1$. Now let us prove the claim. Since $L>\frac d2-1$, we can choose M such that $M>d$ and $M<2L+2$. Take $\nu_m=|Q|^{-\frac12}(1+l(Q)^{-1}|m|)^{-\frac{d+M}{2}}$ and $H_m=\nu_m^{-1}\mathcal{X}_mT^c_{\sigma^2}g$, where $(\mathcal{X}_m)_{m\in\mathbb{Z}^d}$ denotes again the smooth resolution of the unit (1.4), i.e.

$$1=\sum_{m\in\mathbb{Z}^d}\mathcal{X}_m(s),\quad\forall\,s\in\mathbb{R}^d.$$

Applying (5.2), we have

$$\begin{aligned}&\tau\Big(\int_{2Q_{0,m}}|H_m(s)|^2ds\Big)^{\frac12}\\&\lesssim \nu_m^{-1}(1+l(Q)^{-1}|m|)^{-\frac{d+M}{2}}\tau\Big(\int_{(4Q)^c}|T^c_{\sigma^2}g(s)|^2(1+l(Q)^{-1}|s|)^{d+M}ds\Big)^{\frac12}\lesssim 1.\end{aligned}$$

And the normalizing constants ν_m satisfy

$$\begin{aligned}\sum_m|\nu_m|&=|Q|^{-\frac12}\sum_m(1+l(Q)^{-1}|m|)^{-\frac{d+M}{2}}\\&\le |Q|^{-\frac12}\int_{\mathbb{R}^d}(1+l(Q)^{-1}|s|)^{-\frac{d+M}{2}}ds\lesssim 1.\end{aligned}$$

Combining the estimates of $T^c_{\sigma_1}g$ and $T^c_{\sigma_2}g$, we conclude that $\|T^c_\sigma g\|_{\mathrm{h}_1^c}\lesssim 1$. Thus, T^c_σ is bounded on $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$.

Step 3. For the case where $p=1$ and $\alpha\neq 0$, we use the lifting property of Triebel-Lizorkin spaces (see [56, Proposition 3.4]. By the property of the composition of pseudo-differential operators stated in Proposition 2.8, we see that

$$T^c_{\sigma^\alpha}=J^\alpha T^c_\sigma J^{-\alpha}$$

is still a pseudo-differential operator with symbol σ^α in $S^0_{1,\delta}$. Then for any $f\in F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$, we have

$$\|T^c_\sigma f\|_{F_1^{\alpha,c}}=\|J^{-\alpha}T^c_{\sigma^\alpha}J^\alpha f\|_{F_1^{\alpha,c}}\approx\|T^c_{\sigma^\alpha}J^\alpha f\|_{\mathrm{h}_1^c}\lesssim\|J^\alpha f\|_{\mathrm{h}_1^c}\approx\|f\|_{F_1^{\alpha,c}}.$$

Hence, T^c_σ is bounded on $F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$.

Step 4. Finally, we deal with the case $1<p\le\infty$. By the previous steps, $(T^c_\sigma)^*=T^c_{\tilde\sigma}$ is bounded on $F_1^{-\alpha,c}(\mathbb{R}^d,\mathcal{M})$ with $\alpha\in\mathbb{R}$, then it is clear that T^c_σ is bounded on $F_\infty^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$. Given $1<p<\infty$ and $\alpha\in\mathbb{R}$, by interpolation

$$\big(F_\infty^{\alpha,c}(\mathbb{R}^d,\mathcal{M}),F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})\big)_{\frac1p}=F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M}),$$

we get the boundedness of T^c_σ on $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$. □

Remark 5.3. A special case of Theorem 5.1 is that if the symbol is scalar-valued, then

$$\int_{\mathbb{R}^d}\sigma(s,\xi)\widehat f(\xi)e^{2\pi \mathrm{i}s\cdot\xi}d\xi=\int_{\mathbb{R}^d}\widehat f(\xi)\sigma(s,\xi)e^{2\pi \mathrm{i}s\cdot\xi}d\xi.$$

In this case, T^c_σ is also bounded on $\mathrm{h}_p^r(\mathbb{R}^d,\mathcal{M})$ for any $1\le p\le\infty$. By (1.7), we deduce that T^c_σ is bounded on $L_p(\mathcal{N})$.

Corollary 5.4. *Let $n,\alpha\in\mathbb{R}$, $0\le\delta<1$ and $\sigma\in S^n_{1,\delta}$. Then T^c_σ is bounded from $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ to $F_p^{\alpha-n,c}(\mathbb{R}^d,\mathcal{M})$ for any $1\le p\le\infty$.*

Proof. Recall that the Bessel potential of order n maps $F_p^{\alpha,c}$ isomorphically onto $F_p^{\alpha-n,c}$. If $\sigma\in S^n_{1,\delta}$, by Proposition 2.8, we see that

$$\sigma(s,\xi)(1+|\xi|^2)^{-\frac n2}\in S^0_{1,\delta},$$

and its corresponding pseudo-differential operator is $T^c_\sigma J^{-n}$. Since $T^c_\sigma=T^c_\sigma J^{-n}J^n$, the assertion follows obviously from Theorem 5.1. □

6. Forbidden symbols on Triebel-Lizorkin spaces

The purpose of this section is to extend the boundedness results obtained in the previous one to the pseudo-differential operators with forbidden symbols, i.e. the symbols in the class $S^n_{1,1}$. There are two main differences between these operators and those with symbols in $S^n_{1,\delta}$ with $0\le\delta<1$. The first one is that when $\sigma\in S^0_{1,1}$, T^c_σ is not necessarily bounded on $L_2(\mathcal{N})$. The second one is that $S^0_{1,1}$ is not closed under product or adjoint. Fortunately, if the function spaces have a positive degree of smoothness, the operators with symbols in $S^0_{1,1}$ will be bounded on them.

Since for $\sigma\in S^0_{1,1}$, T^c_σ is not necessarily bounded on $L_2(\mathcal{N})$, we cannot expect its boundedness on $L_1\big(\mathcal{M};L_2^c(\mathbb{R}^d)\big)$. However, by Proposition 3.4, we are able to prove its boundedness on

$L_1\big(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\big)$ when $\alpha > 0$. Note that the classical Sobolev space $H_2^\alpha(\mathbb{R}^d)$ is a Hilbert space with the inner product $\langle f, g\rangle = \int_{\mathbb{R}^d} J^\alpha f(s)\overline{J^\alpha g(s)}ds$. By the definition of Hilbert-valued L_p-spaces, we see that $f \in L_1\big(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\big)$ if and only if $J^\alpha f \in L_1\big(\mathcal{M}; L_2^c(\mathbb{R}^d)\big)$.

Lemma 6.1. *Let $\sigma \in S^0_{1,1}$. Then T^c_σ is bounded on $L_1\big(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\big)$ for any $\alpha > 0$.*

Proof. Following the argument for Lemma 5.2 by replacing $(T^c_\sigma)^*$ with $J^\alpha T^c_\sigma$, we see that T^c_σ is bounded on $L_\infty\big(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\big)$. Let $f \in L_1\big(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\big)$. Then f admits the decomposition

$$f = gh,$$

where $\|h\|_{L_1(\mathcal{M})} = \|f\|_{L_1\left(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\right)}$ and $\|g\|_{L_\infty\left(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\right)} = 1$. Indeed, if $A = (\int_{\mathbb{R}^d} |J^\alpha f(s)|^2 ds)^{\frac12}$ is invertible, we could take $g = fA^{-1}$, $h = A$; otherwise we can approximate A by invertable elements in $L_1(\mathcal{M})$, which does not disturb the argument below. From this decomposition, we establish the $L_1\big(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\big)$-norm of $T^c_\sigma(f)$ as follows:

$$\begin{aligned}\|T^c_\sigma(f)\|_{L_1\left(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\right)} &= \|T^c_\sigma(g)h\|_{L_1\left(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\right)}\\ &\le \|T^c_\sigma(g)\|_{L_\infty\left(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\right)}\|h\|_{L_1(\mathcal{M})}\\ &\lesssim \|g\|_{L_\infty\left(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\right)}\|h\|_{L_1(\mathcal{M})}\\ &= \|f\|_{L_1\left(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\right)},\end{aligned}$$

which implies that T^c_σ is bounded on $L_1\big(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\big)$. □

Similar to [53, Theorem 6.3.2], the main theorem in this section also relies on the atomic decomposition. However, the techniques we use are quite different from that of [53]. Apart from the difficulties mentioned in the beginning of section 4, the symbol considered here has the global mapping property, which is more general than the case in [53], i.e. we do not need the assumption that the symbol σ is compactly supported with respect to the second variable.

Theorem 6.2. *Let $\sigma \in S^0_{1,1}$ and $\alpha > 0$. Then T^c_σ is bounded on $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$.*

Proof. Let $f \in F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$. We fix K, L to be two integers such that $K > \alpha + d$ and $L > d$. By the atomic decomposition in Theorem 1.3, f can be written as

$$f = \sum_{j=1}^{\infty}(\mu_j b_j + \lambda_j g_j),$$

where the b_j's are $(\alpha, 1)$-atoms and the g_j's are (α, Q)-atoms, μ_j and λ_j are complex numbers such that

$$\sum_{j=1}^{\infty}(|\mu_j| + |\lambda_j|) \approx \|f\|_{F_1^{\alpha,c}}.$$

In order to prove the assertion, by the above atomic decomposition, it suffices to prove that

$$\|T^c_\sigma b\|_{F_1^{\alpha,c}} \lesssim 1 \quad \text{and} \quad \|T^c_\sigma g\|_{F_1^{\alpha,c}} \lesssim 1,$$

for any $(\alpha, 1)$-atom b and (α, Q)-atom g. We have shown in Corollary 4.6 that

$$\|T^c_\sigma b\|_{F_1^{\alpha,c}} \lesssim 1. \tag{6.1}$$

Thus it remains to consider $T^c_\sigma g$. This is the main part of the proof which will be divided into several steps for clarity.

Step 1. By translation, we may assume that the supporting cube Q of the atom g is centered at the origin. We begin with a split of the symbol σ: Let h_1, h_2 be two nonnegative infinitely differentiable functions on $\mathbb{R}^d$ such that $\operatorname{supp} h_1 \subset (Q)^c$, $\operatorname{supp} h_2 \subset 2Q$ and

$$1 = h_1(\xi) + h_2(\xi), \quad \forall\, \xi \in \mathbb{R}^d.$$

For any $(s, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$, we write

$$\sigma(s,\xi) = h_1(\xi)\sigma(s,\xi) + h_2(\xi)\sigma(s,\xi) \overset{\text{def}}{=} \sigma_1(s,\xi) + \sigma_2(s,\xi).$$

It is clear that σ_1 and σ_2 are still two symbols in $S^0_{1,1}$, and

$$\|T^c_\sigma g\|_{F_1^{\alpha,c}} \leq \|T^c_{\sigma_1} g\|_{F_1^{\alpha,c}} + \|T^c_{\sigma_2} g\|_{F_1^{\alpha,c}}. \tag{6.2}$$

First, we consider the case where the cube Q is of side length one, i.e. $Q = Q_{0,0}$, and deal with the term $\|T^c_{\sigma_1} g\|_{F_1^{\alpha,c}}$ in the above split. Let $(\mathcal{X}_j)_{j\in\mathbb{Z}^d}$ be the resolution of the unit defined in (1.4) and $\widetilde{\mathcal{X}}_j = \mathcal{X}_j(2\cdot)$ for $j \in \mathbb{Z}^d$. We write

$$\begin{aligned} T^c_{\sigma_1} g &= \sum_{\substack{j\in 8Q_{0,0}\\ j\in\mathbb{Z}^d}} T^c_{\sigma_1^j} g + \sum_{\substack{j\notin 8Q_{0,0}\\ j\in\mathbb{Z}^d}} T^c_{\sigma_1^j} g \\ &\overset{\text{def}}{=} G_1 + H_1, \end{aligned} \tag{6.3}$$

where $\sigma_1^j(s,\xi) = \sigma_1(s,\xi)\widetilde{\mathcal{X}}_j(s)$.

We claim that for every $j \in \mathbb{Z}^d$, $T^c_{\sigma_1^j} g$ is the bounded multiple of an $(\alpha, Q_{0,\frac{j}{2}})$-atom (with the convention $Q_{0,\frac{j}{2}} = \frac{j}{2} + Q_{0,0}$). No loss of generality, we prove the claim just for $j = 0$. Applying Lemma 6.1 to the symbol σ_1^0, we get

$$\tau\Big(\int_{\mathbb{R}^d} |J^\alpha T^c_{\sigma_1^0} g(s)|^2 ds\Big)^{\frac12} \lesssim \tau\Big(\int_{\mathbb{R}^d} |J^\alpha g(s)|^2 ds\Big)^{\frac12} \lesssim |Q_{0,0}|^{-\frac12}.$$

Thus, in order to prove the claim, it remains to show that $T^c_{\sigma_1^0} g$ can be written as the linear combination of subatoms and the coefficients satisfy a certain condition. By Definition 1.2, g admits the following representation:

$$g = \sum_{(\mu,l)\leq(0,0)} d_{\mu,l} a_{\mu,l}, \tag{6.4}$$

where the $a_{\mu,l}$'s are $(\alpha, Q_{\mu,l})$-subatoms and the coefficients $d_{\mu,l}$'s are complex numbers satisfying $\sum_{(\mu,l)\leq(0,0)} |d_{\mu,l}|^2 \leq 1$. Then we have

$$T^c_{\sigma_1^0} g = \sum_{(\mu,l)\leq(0,0)} d_{\mu,l} T^c_{\sigma_1^0} a_{\mu,l}.$$

Given $\mu \in \mathbb{N}_0$, let $(\mathcal{X}_{\mu,m})_{m\in\mathbb{Z}^d}$ be a sequence of infinitely differentiable functions on $\mathbb{R}^d$ such that

$$1 = \sum_{m\in\mathbb{Z}^d} \mathcal{X}_{\mu,m}(s), \quad \forall\, s \in \mathbb{R}^d, \tag{6.5}$$

and each $\mathcal{X}_{\mu,0}$ is nonnegative, supported in $2Q_{\mu,0}$ and $\mathcal{X}_{\mu,m}(s) = \mathcal{X}_{\mu,0}(s - 2^{-\mu}m)$. It is the $2^{-\mu}$-dilation of the resolution of the unit in (1.4). We decompose $T^c_{\sigma_1^0} g$ in the following way:

$$T^c_{\sigma_1^0} g = \sum_{\mu=0}^{\infty} \sum_m \mathcal{X}_{\mu,m} \sum_l d_{\mu,l} T^c_{\sigma_1^0} a_{\mu,l}. \tag{6.6}$$

Observe that the only m's that contribute to the above sum $\sum_m$ are those $m \in \mathbb{Z}^d$ such that $2Q_{\mu,m} \cap Q_{0,0} \neq \emptyset$, so $Q_{\mu,m} \subset 2Q_{0,0}$. Thus, we obtain the decomposition

$$T^c_{\sigma_1^0} g = \sum_{(\mu,m)\leq(0,0)} D_{\mu,m} G_{\mu,m}, \tag{6.7}$$

where

$$\begin{aligned} D_{\mu,m} &= \Big(\sum_l |d_{\mu,l}|^2 (1 + |m - l|)^{-(d+1)}\Big)^{\frac12}, \\ G_{\mu,m} &= \frac{1}{D_{\mu,m}} \mathcal{X}_{\mu,m} \sum_l d_{\mu,l} T^c_{\sigma_1^0} a_{\mu,l}. \end{aligned}$$

It is evident that

$$\Big(\sum_{(\mu,m)\leq(0,0)} |D_{\mu,m}|^2\Big)^{\frac12} \lesssim \Big(\sum_{(\mu,l)\leq(0,0)} |d_{\mu,l}|^2\Big)^{\frac12} \leq 1.$$

Now we show that the $G_{\mu,m}$'s are bounded multiple of $(\alpha, Q_{\mu,m})$-subatoms. Firstly, we have $\operatorname{supp} G_{\mu,m} \subset \operatorname{supp} \mathcal{X}_{\mu,m} \subset 2Q_{\mu,m}$. Secondly, by the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\tau\Big(\int_{2Q_{\mu,m}} |\sum_l d_{\mu,l} T^c_{\sigma_1^0} a_{\mu,l}(s)|^2 ds\Big)^{\frac{1}{2}} \\
\lesssim\ &\Big(\sum_l |d_{\mu,l}|^2 (1+|m-l|)^{-(d+1)}\Big)^{\frac{1}{2}} \\
&\cdot \sum_l (1+|m-l|)^{\frac{1-M}{2}} \tau\Big(\int_{2Q_{\mu,m}} (1+2^\mu(s-2^{-\mu}l))^{d+M} |T^c_{\sigma_1^0} a_{\mu,l}(s)|^2 ds\Big)^{\frac{1}{2}}.
\end{aligned} \tag{6.8}
$$

If we take $M = 2L+1$, since $L > d$, we have $\frac{1-M}{2} < -d$. Applying Lemma 4.1, we get

$$
\tau\Big(\int_{\mathbb{R}^d} |G_{\mu,m}(s)|^2 ds\Big)^{\frac{1}{2}} \lesssim \sum_l (1+|m-l|)^{\frac{1-M}{2}} |Q_{\mu,l}|^{\frac{\alpha}{d}} \lesssim |Q_{\mu,m}|^{\frac{\alpha}{d}}.
$$

Similarly, the derivative estimates in Lemma 4.1 ensure that

$$
\tau\Big(\int |D^\gamma G_{\mu,m}(s)|^2 ds\Big)^{\frac{1}{2}} \lesssim |Q_{\mu,m}|^{\frac{\alpha}{d} - \frac{|\gamma|_1}{d}}, \quad \forall\, |\gamma|_1 \le [\alpha]+1.
$$

Since $\alpha > 0$, no moment cancellation for subatoms is required. Thus, we have proved that the $G_{\mu,m}$'s are bounded multiple of $(\alpha, Q_{\mu,m})$-subatoms, then the claim is proved. Therefore, G_1 in (6.3) is the finite sum of $(\alpha, Q_{0,j})$-atoms, which yields $\|G_1\|_{F_1^{\alpha,c}} \lesssim 1$ by Theorem 1.3.

The term H_1 in (6.3) is much easier to handle. Observe that H_1 corresponds to the symbol $\sigma(s,\xi) \sum_{j \notin 8Q_{0,0}} \widetilde{\mathcal{X}}_j(s)$, whose s-support is in $(6Q_{0,0})^c$. Thus, we apply Corollary 4.4 directly to get that

$$
\|H_1\|_{F_1^{\alpha,c}} \lesssim 1.
$$

Step 2. Let us consider now the case where the supporting cube Q of g has side length less than one. As above, we may still assume that Q is centered at the origin. Let g be an $(\alpha, Q_{k,0})$-atom with $k \in \mathbb{N}$. Then g is given by

$$
g = \sum_{(\mu,l) \le (k,0)} d_{\mu,l} a_{\mu,l} \quad \text{with} \sum_{(\mu,l)} |d_{\mu,l}|^2 \le |Q_{k,0}|^{-1} = 2^{kd}.
$$

We normalize g as

$$
\begin{aligned}
h &= 2^{k(\alpha-d)} g(2^{-k}\cdot) \\
&= \sum_{(\mu,l)\le(k,0)} 2^{-\frac{kd}{2}} d_{\mu,l} 2^{k(\alpha-\frac{d}{2})} a_{\mu,l}(2^{-k}\cdot) \\
&= \sum_{(\mu,l)\le(k,0)} \widetilde{d}_{\mu,l} \widetilde{a}_{\mu,l},
\end{aligned}
$$

where $\widetilde{a}_{\mu,l} = 2^{k(\alpha-\frac{d}{2})} a_{\mu,l}(2^{-k}\cdot)$ and $\widetilde{d}_{\mu,l} = 2^{-\frac{kd}{2}} d_{\mu,l}$. Then it is easy to see that each $\widetilde{a}_{\mu,l}$ is an $(\alpha, Q_{\mu-k,l})$-subatom and thus h is an $(\alpha, Q_{0,0})$-atom. Define $\sigma_{1,k}(s,\xi) = \sigma_1(2^{-k}s, 2^k\xi)$, then we have

$$
\begin{aligned}
T^c_{\sigma_1} g(s) &= \int_{\mathbb{R}^d} \sigma_1(s,\xi) \widehat{g}(\xi) e^{2\pi i s\cdot\xi} d\xi \\
&= 2^{-k\alpha} \int_{\mathbb{R}^d} \sigma_1(s,\xi) \widehat{h}(2^{-k}\xi) e^{2\pi i s\cdot\xi} d\xi \\
&= 2^{k(d-\alpha)} \int_{\mathbb{R}^d} \sigma_{1,k}(2^k s,\xi) \widehat{h}(\xi) e^{2\pi i 2^k s\cdot\xi} d\xi \\
&= 2^{k(d-\alpha)} T^c_{\sigma_{1,k}} h(2^k s).
\end{aligned} \tag{6.9}
$$

Since the ξ-support of σ_1 is away from the origin, we have

$$
\|D_s^\gamma D_\xi^\beta \sigma_{1,k}(s,\xi)\|_{\mathcal{M}} \le C_{\gamma,\beta} |\xi|^{|\gamma|_1 - |\beta|_1} \approx C_{\gamma,\beta}(1+|\xi|)^{|\gamma|_1-|\beta|_1}, \quad \forall\, k \in \mathbb{N}.
$$

Thus, $\sigma_{1,k}$ is still a symbol in the class $S^0_{1,1}$. Then, applying the result for $(\alpha, Q_{0,0})$-atoms obtained in Step 1 to the symbol $\sigma_{1,k}$, we get $\|T^c_{\sigma_{1,k}} h\|_{F_1^{\alpha,c}} \lesssim 1$. In order to return back to the $F_1^{\alpha,c}$-norm of $T^c_{\sigma_1}g$, by (6.9), we need a dilation argument. Since $\alpha>0$, we can invoke the characterization of $F_1^{\alpha,c}$-norm in [56, Corollary 3.10]:

$$\|f\|_{F_1^{\alpha,c}} \approx \|f\|_1 + \Big\|\big(\int_0^\infty \varepsilon^{-2\alpha}|\varphi_\varepsilon * f|^2 \frac{d\varepsilon}{\varepsilon}\big)^{\frac12}\Big\|_1,$$

where $\varphi_\varepsilon = \mathcal{F}^{-1}(\varphi(\varepsilon\cdot))$. For $\lambda>0$, we have $\|f(\lambda\cdot)\|_1 = \lambda^{-d}\|f\|_1$, and

$$\Big\|\big(\int_0^\infty \varepsilon^{-2\alpha}|\varphi_\varepsilon * f(\lambda\cdot)|^2 \frac{d\varepsilon}{\varepsilon}\big)^{\frac12}\Big\|_1 = \lambda^{\alpha-d}\Big\|\big(\int_0^\infty \varepsilon^{-2\alpha}|\varphi_\varepsilon * f|^2 \frac{d\varepsilon}{\varepsilon}\big)^{\frac12}\Big\|_1$$

since $(\varphi_\varepsilon * f(\lambda\cdot))(s) = \varphi_{\lambda\varepsilon} * f(\lambda s)$. Taking $\lambda = 2^k$, we deduce

$$\begin{aligned}
\|T^c_{\sigma_1}g\|_{F_1^{\alpha,c}} &\approx \|T^c_{\sigma_1}g\|_1 + \Big\|\big(\int_0^\infty \varepsilon^{-2\alpha}|\varphi_\varepsilon * T^c_{\sigma_1}g|^2 \frac{d\varepsilon}{\varepsilon}\big)^{\frac12}\Big\|_1\\
&= 2^{k(d-\alpha)}\Big(\|T^c_{\sigma_{1,k}}h(2^k\cdot)\|_1 + \Big\|\big(\int_0^\infty \varepsilon^{-2\alpha}|\varphi_\varepsilon * T^c_{\sigma_{1,k}}h(2^k\cdot)|^2 \frac{d\varepsilon}{\varepsilon}\big)^{\frac12}\Big\|_1\Big)\\
&= 2^{k(d-\alpha)}\Big(2^{-kd}\|T^c_{\sigma_{1,k}}h\|_1 + 2^{k(\alpha-d)}\Big\|\big(\int_0^\infty \varepsilon^{-2\alpha}|\varphi_\varepsilon * T^c_{\sigma_{1,k}}h|^2 \frac{d\varepsilon}{\varepsilon}\big)^{\frac12}\Big\|_1\Big)\\
&\lesssim \|T^c_{\sigma_{1,k}}h\|_{F_1^{\alpha,c}}.
\end{aligned}$$

This ensures

$$\|T^c_{\sigma_1}g\|_{F_1^{\alpha,c}} \lesssim \|T^c_{\sigma_{1,k}}h\|_{F_1^{\alpha,c}} \lesssim 1.$$

Step 3. It remains to deal with the term with symbol σ_2 in (6.2). Note that $\sigma_2 = h_2(\xi)\sigma(s,\xi)$ with $\sigma\in S^0_{1,1}$ and $\operatorname{supp} h_2 \in 2Q$. Then for $\delta<1$, say $\delta=\frac{9}{10}$, we have $\sigma_2\in S^0_{1,\delta}$. Indeed, by definition, we have, for every $s\in\mathbb{R}$,

$$\begin{aligned}
\|D_s^\gamma D_\xi^\beta \sigma_2(s,\xi)\|_{\mathcal{M}} &\lesssim \sum_{\beta_1+\beta_2=\beta} \|D_s^\gamma D_\xi^{\beta_1}\sigma(s,\xi)\cdot D^{\beta_2}h_2(\xi)\|_{\mathcal{M}}\\
&\le \sum_{\beta_1+\beta_2=\beta} C_{\gamma,\beta_1}(1+|\xi|)^{|\gamma|_1-|\beta_1|_1}\cdot|D^{\beta_2}h_2(\xi)|.
\end{aligned}$$

But since h_2 is an infinitely differentiable function with support $2Q$, it is clear that for $\xi\in 2Q$,

$$(1+|\xi|)^{|\gamma|_1-|\beta_1|_1} \le C_\gamma(1+|\xi|)^{\frac{9}{10}|\gamma|_1-|\beta_1|_1}, \quad\text{and}\quad |D^{\beta_2}h_2(\xi)| \le C_{\beta_2}(1+|\xi|)^{-|\beta_2|_1}.$$

Putting these two inequalities into the estimate of $\|D_s^\gamma D_\xi^\beta\sigma_2(s,\xi)\|_{\mathcal{M}}$, we obtain

$$\|D_s^\gamma D_\xi^\beta \sigma_2(s,\xi)\|_{\mathcal{M}} \le C_{\gamma,\beta}(1+|\xi|)^{\frac{9}{10}|\gamma|_1-|\beta|_1},$$

which yields $\sigma_2\in S^0_{1,\frac{9}{10}}$. Therefore, it follows from Theorem 5.1 that $\|T^c_{\sigma_2}g\|_{F_1^{\alpha,c}} \lesssim \|g\|_{F_1^{\alpha,c}}$ for $g\in F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$. Combining this with the estimates in the first two steps, we complete the proof of the theorem. □

If $\sigma\in S^0_{1,1}$, it is not true in general that $(T^c_\sigma)^*$ corresponds to a symbol in the class $S^0_{1,1}$. However, if we assume additionally this last condition, duality and interpolation will give the following boundedness of T^c_σ:

Theorem 6.3. *Let $1<p<\infty$ and $\sigma\in S^0_{1,1}$, $\alpha\in\mathbb{R}$. If $(T^c_\sigma)^*$ admits a symbol in the class $S^0_{1,1}$, then T^c_σ is bounded on $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$.*

A similar argument as in the proof of Corollary 5.4 gives the following results concerning the symbols in $S^n_{1,1}$ with $n\in\mathbb{R}$.

Corollary 6.4. *Let $n\in\mathbb{R}$, $\sigma\in S^n_{1,1}$ and $\alpha>0$. If $\alpha>n$, then T^c_σ is bounded from $F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ to $F_1^{\alpha-n,c}(\mathbb{R}^d,\mathcal{M})$.*

Corollary 6.5. *Let n,α and σ be the same as above, and $1<p<\infty$. If $(T^c_\sigma)^*$ admits a symbol in the class $S^n_{1,1}$, then T^c_σ is bounded from $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ to $F_p^{\alpha-n,c}(\mathbb{R}^d,\mathcal{M})$.*

7. Applications

The main target of this section is to apply the results obtained previously to pseudo-differential operators over quantum tori. Our strategy is to transfer this problem by the transference method introduced in [37] to the operator-valued pseudo-differential operators on the usual torus $\mathbb{T}^d$. Let us begin with the latter case by a periodization argument.

7.1. **Applications to tori.**

In this subsection, $\mathcal{M}$ still denotes a von Neumann algebra with a normal semifinite faithful trace τ, but $\mathcal{N} = L_\infty(\mathbb{T}^d)\overline{\otimes}\mathcal{M}$.

We identify $\mathbb{T}^d$ with the unit cube $\mathbb{I}^d = [0,1)^d$ via $(e^{2\pi \mathrm{i} s_1}, \cdots, e^{2\pi \mathrm{i} s_d}) \leftrightarrow (s_1, \cdots, s_d)$. Under this identification, multiplication in $\mathbb{T}^d$ corresponds to the usual coordinatewise addition modulo 1 in $\mathbb{I}^d$, i.e. when $z = (e^{2\pi \mathrm{i} s_1}, \cdots, e^{2\pi \mathrm{i} s_d}) \leftrightarrow (s_1, \cdots, s_d)$ and $\omega = (e^{2\pi \mathrm{i} t_1}, \cdots, e^{2\pi \mathrm{i} t_d}) \leftrightarrow (t_1, \cdots, t_d)$, $z\omega^{-1} \in \mathbb{T}^d$ is identified with $s - t \in \mathbb{I}^d$ modulo 1. An interval of $\mathbb{I}^d$ is either a subinterval of $\mathbb{I}$ or a union $[b,1] \cup [0,a]$ with $0 < a < b < 1$, the latter union being the interval $[b-1,a]$ of $\mathbb{I}$ (modulo 1). So the cubes of $\mathbb{I}^d$ are exactly those of $\mathbb{T}^d$. Accordingly, functions on $\mathbb{T}^d$ and $\mathbb{I}^d$ are identified too.

Recall that φ is a Schwartz function satisfying (1.1). Then for every $m \in \mathbb{Z}^d \setminus \{0\}$,

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}m) = \sum_{j\geq 0} \varphi(2^{-j}m) = 1.$$

This tells us that in the torus case $\{\varphi(2^{-j}\cdot)\}_{j\geq 0}$ gives a resolvent of the unit. According to this, we make a slight change of the notation that we used before:

$$\varphi^{(j)} = \varphi(2^{-j}\cdot),\ \forall\, j \geq 0.$$

Let $\varphi_j = \mathcal{F}^{-1}(\varphi^{(j)})$ for any $j \geq 0$. Now we periodize φ_j as

$$\widetilde{\varphi}_j(z) = \sum_{m\in\mathbb{Z}^d} \varphi_j(s+m) \quad \text{with} \quad z = (e^{2\pi \mathrm{i} s_1}, \dots, e^{2\pi \mathrm{i} s_d}),\ s = (s_1, \cdots, s_d).$$

Then, we can easily see that $\widetilde{\varphi}_j$ admits the following Fourier series:

$$\widetilde{\varphi}_j(z) = \sum_{m\in\mathbb{Z}^d} \varphi(2^{-j}m) z^m. \tag{7.1}$$

Thus, for any $f \in \mathcal{S}'(\mathbb{T}^d; L_1(\mathcal{M}) + \mathcal{M})$, whenever it exists,

$$\widetilde{\varphi}_j * f(z) = \int_{\mathbb{T}^d} \widetilde{\varphi}_j(zw^{-1}) f(w)dw = \sum_{m\in\mathbb{Z}^d} \varphi(2^{-j}m)\widehat{f}(m)z^m \quad z \in \mathbb{T}^d.$$

The following definition was given in [59, Section 4.5].

Definition 7.1. *Let $1 \leq p < \infty$ and $\alpha \in \mathbb{R}^d$. The column operator-valued Triebel-Lizorkin space $F_p^{\alpha,c}(\mathbb{T}^d, \mathcal{M})$ is defined to be*

$$F_p^{\alpha,c}(\mathbb{T}^d, \mathcal{M}) = \{f \in \mathcal{S}'(\mathbb{T}^d; L_1(\mathcal{M}) + \mathcal{M}) : \|f\|_{F_p^{\alpha,c}} < \infty\},$$

where

$$\|f\|_{F_p^{\alpha,c}} = \|\widehat{f}(0)\|_{L_p(\mathcal{M})} + \big\|(\sum_{j\geq 0} 2^{2j\alpha}|\widetilde{\varphi}_j * f|^2)^{\frac{1}{2}}\big\|_{L_p(\mathcal{N})}.$$

The row and mixture spaces $F_p^{\alpha,r}(\mathbb{T}^d, \mathcal{M})$ and $F_p^{\alpha}(\mathbb{T}^d, \mathcal{M})$, and the corresponding spaces for $p = \infty$ are defined similarly to the Euclidean case.

By the discussion before (7.1), if we identify a function f on $\mathbb{T}^d$ as a 1-periodic function f_{pe} on $\mathbb{R}^d$, then the convolution $\widetilde{\varphi}_j * f$ on $\mathbb{T}^d$ coincides with the convolution $\varphi_j * f_{\mathrm{pe}}$ on $\mathbb{R}^d$. More precisely:

$$\widetilde{\varphi}_j * f(z) = \varphi_j * f_{\mathrm{pe}}(s) \quad \text{with} \quad z = (e^{2\pi \mathrm{i} s_1}, \cdots, e^{2\pi \mathrm{i} s_d}).$$

By the almost orthogonality of the Littlewood-Paley decomposition given in (1.3), we get the following easy equivalent norm of $F_p^{\alpha,c}(\mathbb{I}^d, \mathcal{M})$:

$$\|f_{\mathrm{pe}}\|_{F_p^{\alpha,c}(\mathbb{I}^d,\mathcal{M})} \approx \|\phi_0 * f_{\mathrm{pe}}\|_p + \big\|(\sum_{j\geq 0} 2^{2j\alpha}|\varphi_j * f_{\mathrm{pe}}(z)|^2)^{\frac{1}{2}}\big\|_p,$$

where $\widehat{\phi}_0(\xi) = 1 - \sum_{j\geq 0} \varphi(2^{-j}\xi)$. Since $\widehat{\phi}_0$ is supported in $\{\xi : |\xi| \leq 1\}$ and $\widehat{\phi}_0(\xi) = 1$ if $|\xi| \leq \frac{1}{2}$, it then follows that

$$\|\phi_0 * f_{\rm pe}\|_p = \|\widehat{f}(0)\|_p \,.$$

Hence, combining the estimates above, we have

$$\|f\|_{F_p^{\alpha,c}(\mathbb{T}^d,\mathcal{M})} \approx \|f_{\rm pe}\|_{F_p^{\alpha,c}(\mathbb{I}^d,\mathcal{M})}. \tag{7.2}$$

Thus $F_p^{\alpha,c}(\mathbb{T}^d,\mathcal{M})$ embeds into $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ isomorphically. The equivalence (7.2) allows us to reduce the treatment of $\mathbb{T}^d$ to that of $\mathbb{R}^d$; and by periodicity, all the functions considered now are restricted on $\mathbb{I}^d$. We are not going to state the properties of $F_p^{\alpha,c}(\mathbb{T}^d,\mathcal{M})$ specifically, and refer the reader to [59, Section 4.5] for similar results on quantum torus.

We also briefly introduce operator-valued Sobolev and Besov spaces on $\mathbb{T}^d$. Let $1 \leq p, q < \infty$ and $\alpha \in \mathbb{R}^d$. The potential Sobolev space $H_p^\alpha(\mathbb{T}^d; L_p(\mathcal{M}))$ is defined as

$$H_p^\alpha(\mathbb{T}^d; L_p(\mathcal{M})) = \{f \in \mathcal{S}'(\mathbb{T}^d; L_1(\mathcal{M}) + \mathcal{M}) : \|f\|_{H_p^\alpha} := \|J^\alpha f\|_p < \infty\},$$

where J^α denotes the α-order Bessel potential on $\mathbb{T}^d$. The Besov space $B_{p,q}^\alpha(\mathbb{T}^d; L_p(\mathcal{M}))$ is defined as

$$B_{p,q}^\alpha(\mathbb{T}^d; L_p(\mathcal{M})) = \{f \in \mathcal{S}'(\mathbb{T}^d; L_1(\mathcal{M}) + \mathcal{M}) : \|f\|_{B_{p,q}^\alpha} < \infty\},$$

where

$$\|f\|_{B_{p,q}^\alpha} = \|\widehat{f}(0)\|_{L_p(\mathcal{M})} + \Big(\sum_{j\geq 0} 2^{qj\alpha}\|\widetilde{\varphi}_j * f\|_p^q\Big)^{\frac{1}{q}}.$$

For a fixed $1 \leq p \leq \infty$, these spaces are the Banach-valued Sobolev and Besov spaces studied in [2], the Banach space being $L_p(\mathcal{M})$. In analogy to (7.2), we have

$$\|f\|_{H_p^\alpha(\mathbb{T}^d;L_p(\mathcal{M}))} \approx \|f_{\rm pe}\|_{H_p^\alpha(\mathbb{I}^d;L_p(\mathcal{M}))} \tag{7.3}$$

and

$$\|f\|_{B_{p,q}^\alpha(\mathbb{T}^d;L_p(\mathcal{M}))} \approx \|f_{\rm pe}\|_{B_{p,q}^\alpha(\mathbb{I}^d;L_p(\mathcal{M}))}. \tag{7.4}$$

Let us turn to the study of toroidal symbols. In the discrete case, the derivatives degenerate into discrete difference operators. Let $\sigma : \mathbb{Z}^d \to \mathcal{M}$. For $1 \leq j \leq d$, let e_j be the j-th canonical basis of $\mathbb{R}^d$. We define the forward and backward partial difference operators Δ_{m_j} and $\overline{\Delta}_{m_j}$:

$$\Delta_{m_j}\sigma(m) := \sigma(m + e_j) - \sigma(m), \quad \overline{\Delta}_{m_j}\sigma(m) := \sigma(m) - \sigma(m - e_j),$$

and for any $\beta \in \mathbb{N}_0^d$,

$$\Delta_m^\beta := \Delta_{m_1}^{\beta_1} \cdots \Delta_{m_d}^{\beta_d}, \quad \overline{\Delta}_m^\beta := \overline{\Delta}_{m_1}^{\beta_1} \cdots \overline{\Delta}_{m_d}^{\beta_d}.$$

Definition 7.2. *Let $0 \leq \delta, \rho \leq 1$ and $\gamma, \beta \in \mathbb{N}_0^d$. Then the toroidal symbol class $S_{\rho,\delta}^n(\mathbb{T}^d \times \mathbb{Z}^d)$ consists of those $\mathcal{M}$-valued functions $\sigma(s, m)$ which are smooth in s for all $m \in \mathbb{Z}^d$, and satisfy*

$$\|D_s^\gamma \Delta_m^\beta \sigma(s, m)\|_{\mathcal{M}} \leq C_{\gamma,\beta,m}(1 + |m|)^{n-\rho|\beta|_1+\delta|\gamma|_1} \text{ for all } \gamma, \beta \in \mathbb{N}_0^d.$$

Definition 7.3. *Let $\sigma \in S_{\rho,\delta}^n(\mathbb{T}^d \times \mathbb{Z}^d)$. For any $f \in \mathcal{S}'(\mathbb{T}^d; L_1(\mathcal{M}))$, we define the corresponding toroidal pseudo-differential operator as follows:*

$$T_\sigma^c f(s) = \sum_{m\in\mathbb{Z}^d} \sigma(s, m)\widehat{f}(m)e^{2\pi i s\cdot m}.$$

When studying the toroidal pseudo-differential operators T_σ^c on $\mathbb{T}^d$, especially its action on operator-valued Triebel-Lizorkin spaces on $\mathbb{T}^d$, a very useful tool is to extend the toroidal symbol to a symbol defined on $\mathbb{T}^d \times \mathbb{R}^d$, which reduces the torus case to the Euclidean one. This allows us to apply the results in the last sections. The extension of scalar-valued toroidal symbol has been well studied in [45]. With some minor modifications, the arguments used in [45] can be adjusted to our operator-valued setting.

The following lemma is taken from [45]. Denote by $\delta_0(\xi)$ the Kronecker delta function at 0, i.e., $\delta_0(0) = 1$ and $\delta_0(\xi) = 0$ if $\xi \neq 0$.

Lemma 7.4. *For each $\beta \in \mathbb{N}_0^d$, there exists a function $\phi_\beta \in \mathcal{S}(\mathbb{R}^d)$ and a function $\zeta \in \mathcal{S}(\mathbb{R}^d)$ such that*

$$\sum_{k\in\mathbb{Z}^d} \zeta(s+k) \equiv 1,$$

$$\widehat{\zeta}\,|_{\mathbb{Z}^d}(\xi) = \delta_0(\xi) \quad \text{and} \quad D_\xi^\beta(\widehat{\zeta})(\xi) = \overline{\Delta}_\xi^\beta \phi_\beta(\xi),$$

for any $\xi \in \mathbb{R}^d$.

Now let us give the operator-valued analogue of Theorem 4.5.3 in [45].

Lemma 7.5. *Let $0 \le \rho, \delta \le 1$ and $n \in \mathbb{R}$. A symbol $\sigma \in S^n_{\rho,\delta}(\mathbb{T}^d \times \mathbb{Z}^d)$ is a toroidal symbol if and only if there exists an Euclidean symbol $\widetilde{\sigma} \in S^n_{\rho,\delta}(\mathbb{T}^d \times \mathbb{R}^d)$ such that $\sigma = \widetilde{\sigma}\,|_{\mathbb{T}^d\times\mathbb{Z}^d}$.*

Proof. We first prove the "if" part. Let $\widetilde{\sigma} \in S^n_{\rho,\delta}(\mathbb{T}^d \times \mathbb{R}^d)$. If $|\beta|_1 = 1$, then by the mean value theorem for vector-valued functions, we have

$$\|\Delta_m^\beta D_s^\gamma \sigma(s,m)\|_{\mathcal{M}} \le \sup_{0\le\theta\le 1} \big\|\partial_\xi^\beta D_s^\gamma \widetilde{\sigma}(s, m+\theta\beta)\big\|_{\mathcal{M}}.$$

For a general multi-index $\beta \in \mathbb{N}_0^d$, we use induction. Writing $\beta = \beta' + \delta_j$ and using the induction hypothesis, we get

$$\begin{aligned}
\|\Delta_m^\beta D_s^\gamma \sigma(s,m)\|_{\mathcal{M}} &= \|\Delta_m^{\delta_j}(\Delta_m^{\beta'} D_s^\gamma \widetilde{\sigma}(s,m))\| \\
&\le \sup_{0\le\theta\le1} \|\partial_j(\Delta_m^{\beta'} D_s^\gamma \widetilde{\sigma}(s, m+\theta\delta_j))\|_{\mathcal{M}} \\
&= \sup_{0\le\theta\le1} \|\Delta_m^{\beta'}(\partial_j D_s^\gamma \widetilde{\sigma}(s, m+\theta\delta_j))\|_{\mathcal{M}} \\
&\le \sup_{0\le\theta'\le1} \|D_\xi^{\beta'} \partial_j D_s^\gamma \widetilde{\sigma}(s, m+\theta'\beta)\|_{\mathcal{M}} \\
&= \sup_{0\le\theta'\le1} \|D_\xi^{\beta} D_s^\gamma \widetilde{\sigma}(s, m+\theta'\beta)\|_{\mathcal{M}}.
\end{aligned}$$

Thus we deduce that

$$\begin{aligned}
\|\Delta_m^\beta D_s^\gamma \sigma(s,m)\|_{\mathcal{M}} &\le \sup_{0\le\theta'\le1} \|D_\xi^{\beta} D_s^\gamma \widetilde{\sigma}(s, m+\theta'\beta)\|_{\mathcal{M}} \\
&\le C'_{\alpha,\beta,m}(1+|m|)^{n-\rho|\beta|_1+\delta|\gamma|_1}.
\end{aligned}$$

Now let us show the "only if" part. In the proof of Theorem 4.5.3 in [45], the desired Euclidean symbol is constructed with the help of the functions in Lemma 7.4. We can transfer directly the arguments in [45] to our setting. But we still include a proof for completeness. Let $\zeta \in \mathcal{S}(\mathbb{R}^d)$ be as in Lemma 7.4. Define a function $\widetilde{\sigma} : \mathbb{T}^d \times \mathbb{R}^d \to \mathcal{M}$ by

$$\widetilde{\sigma}(s,\xi) = \sum_{m\in\mathbb{Z}^d} \widehat{\zeta}(\xi - m)\sigma(s,m).$$

Thus, $\sigma = \widetilde{\sigma}\,|_{\mathbb{T}^d\times\mathbb{Z}^d}$. Moreover, using summation by parts, we have

$$\begin{aligned}
\|D_s^\gamma D_\xi^\beta \widetilde{\sigma}(s,\xi)\|_{\mathcal{M}} &= \big\|\sum_{m\in\mathbb{Z}^d} D_\xi^\beta \widehat{\zeta}(\xi-m) D_s^\beta \sigma(s,m)\big\|_{\mathcal{M}} \\
&= \big\|\sum_{m\in\mathbb{Z}^d} \overline{\Delta}_\xi^\beta \phi_\beta(\xi-m) D_s^\gamma \sigma(s,m)\big\|_{\mathcal{M}} \\
&= \|(-1)^{|\beta|_1} \sum_{m\in\mathbb{Z}^d} \phi_\beta(\xi-m) \Delta_m^\beta D_s^\gamma \sigma(s,m)\|_{\mathcal{M}} \\
&\lesssim \sum_{m\in\mathbb{Z}^d} |\phi_\beta(\xi-m)|(1+|m|)^{n-\rho|\beta|_1+\delta|\beta|_1} \\
&\lesssim \sum_{m\in\mathbb{Z}^d} |\phi_\beta(\xi-m)|(1+|\xi-m|)^{n-\rho|\beta|_1+\delta|\gamma|_1}(1+|\xi|)^{n-\rho|\beta|_1+\delta|\gamma|_1} \\
&\lesssim (1+|\xi|)^{n-\rho|\beta|_1+\delta|\gamma|_1},
\end{aligned}$$

whence, $\widetilde{\sigma} \in S^n_{\rho,\delta}(\mathbb{T}^d \times \mathbb{R}^d)$. $\square$

Theorem 7.6. *Let $\sigma \in S^0_{1,\delta}(\mathbb{T}^d \times \mathbb{Z}^d)$.*

i) *If $0 \le \delta < 1$ and $\alpha \in \mathbb{R}$, then T^c_σ is bounded on $H^\alpha_2(\mathbb{T}^d; L_2(\mathcal{M}))$, and bounded on $B^\alpha_{p,q}(\mathbb{T}^d; L_p(\mathcal{M}))$ for $1 \le p, q \le \infty$.*
ii) *If $\delta = 1$ and $\alpha > 0$, then T^c_σ is bounded on $H^\alpha_2(\mathbb{T}^d; L_2(\mathcal{M}))$, and bounded on $B^\alpha_{p,q}(\mathbb{T}^d; L_p(\mathcal{M}))$ for $1 \le p, q \le \infty$.*

Proof. By Lemma 7.5, there exists $\widetilde{\sigma}$ in $S^n_{\rho,\delta}(\mathbb{T}^d \times \mathbb{R}^d)$ such that $\sigma = \widetilde{\sigma}\mid_{\mathbb{T}^d \times \mathbb{Z}^d}$. Let $f \in \mathcal{S}'(\mathbb{T}^d, L_1(\mathcal{M}))$. By the identification $\mathbb{T}^d \approx \mathbb{I}^d$, for any $z \in \mathbb{T}^d$, there exists $s \in \mathbb{I}^d$ such that

$$\begin{aligned} T^c_\sigma f(z) &= \sum_{m\in\mathbb{Z}^d} \sigma(s,m)\widehat{f}(m)e^{2\pi \mathrm{i} s\cdot m} \\ &= \int_{\mathbb{R}^d} \widetilde{\sigma}(s,\xi)\widehat{f}_{\mathrm{pe}}(\xi)e^{2\pi \mathrm{i} s\cdot\xi}d\xi = T^c_{\widetilde{\sigma}} f_{\mathrm{pe}}(s). \end{aligned}$$

Now we apply Corollary 3.3, Proposition 3.4 and Theorem 3.5 to the symbol $\widetilde{\sigma}$ and f_{pe}. Then by (7.3) and (7.4), we get the desired conclusions. □

Theorem 7.7. *Let $\sigma \in S^0_{1,\delta}(\mathbb{T}^d \times \mathbb{Z}^d)$ and $\alpha \in \mathbb{R}$.*

i) *If $0 \le \delta < 1$, then T^c_σ is bounded on $F^{\alpha,c}_p(\mathbb{T}^d, \mathcal{M})$ for every $1 \le p \le \infty$.*
ii) *If $\delta = 1$ and $\alpha > 0$, then T^c_σ is bounded on $F^{\alpha,c}_1(\mathbb{T}^d, \mathcal{M})$.*
iii) *If $\delta = 1$ and $(T^c_\sigma)^*$ admits a symbol in the class $S^0_{1,1}(\mathbb{T}^d \times \mathbb{Z}^d)$, then T^c_σ is bounded on $F^{\alpha,c}_p(\mathbb{T}^d, \mathcal{M})$ for any $1 < p < \infty$.*

Proof. The proof of this theorem is similar to that of the last one. This time we appeal to Theorems 5.1, 6.2 and 6.3 and the equivalence (7.2). □

7.2. **Applications to quantum tori.** We now apply the above results to the quantum case. To this end, we briefly recall the relevant definitions, and refer the reader to [11] and [59] for more details. Let $d \ge 2$ and $\theta = (\theta_{kj})$ be a real skew symmetric $d \times d$-matrix. The associated d-dimensional noncommutative torus $\mathcal{A}_\theta$ is the universal C^*-algebra generated by d unitary operators $U_1, \dots, U_d$ satisfying the following commutation relation

$$U_k U_j = e^{2\pi \mathrm{i}\theta_{kj}} U_j U_k, \quad j,k = 1,\dots,d.$$

We will use standard notation from multiple Fourier series. Let $U = (U_1, \cdots, U_d)$. For $m = (m_1, \cdots, m_d) \in \mathbb{Z}^d$, define

$$U^m = U_1^{m_1} \cdots U_d^{m_d}.$$

A polynomial in U is a finite sum

$$x = \sum_{m\in\mathbb{Z}^d} \alpha_m U^m \quad \text{with} \quad \alpha_m \in \mathbb{C}.$$

The involution algebra $\mathcal{P}_\theta$ of all such polynomials is dense in $\mathcal{A}_\theta$. For any polynomial x as above, we define

$$\tau(x) = \alpha_0.$$

Then τ extends to a faithful tracial state τ on $\mathcal{A}_\theta$. Let $\mathbb{T}^d_\theta$ be the w^*-closure of $\mathcal{A}_\theta$ in the GNS representation of τ. This is our d-dimensional quantum torus. The state τ extends to a normal faithful tracial state on $\mathbb{T}^d_\theta$ that will be denoted again by τ. Note that if $\theta = 0$, then $\mathbb{T}^d_\theta = L_\infty(\mathbb{T}^d)$ and τ coincides with the integral on $\mathbb{T}^d$ against normalized Haar measure dz.

Any $x \in L_1(\mathbb{T}^d_\theta)$ admits a formal Fourier series:

$$x \sim \sum_{m\in\mathbb{Z}^d} \widehat{x}(m)U^m \ \text{ with } \ \widehat{x}(m) = \tau((U^m)^*x).$$

In [37], a transference method has been introduced to overcome the full noncommutativity of quantum tori and to use methods of operator-valued harmonic analysis. Let $\mathcal{N}_\theta = L_\infty(\mathbb{T}^d)\overline{\otimes}\mathbb{T}^d_\theta$, equipped with the tensor trace $\nu = \int dz \otimes \tau$. For each $z \in \mathbb{T}^d$, define π_z to be the isomorphism of $\mathbb{T}^d_\theta$ determined by

$$\pi_z(U^m) = z^m U^m = z_1^{m_1} \cdots z_d^{m_d} U_1^{m_1} \cdots U_d^{m_d}. \tag{7.5}$$

This isomorphism preserves the trace τ. Thus for every $1 \le p < \infty$,

$$\|\pi_z(x)\|_p = \|x\|_p, \; \forall\, x \in L_p(\mathbb{T}^d_\theta).$$

The main points of the transference method are contained in the following lemma from [11].

Lemma 7.8. i) *Let $1 \le p \le \infty$. For any $x \in L_p(\mathbb{T}^d_\theta)$, the function $\widetilde{x} : z \mapsto \pi_z(x)$ is continuous from $\mathbb{T}^d$ to $L_p(\mathbb{T}^d_\theta)$ (with respect to the w^*-topology for $p = \infty$).*

ii) *If $x \in L_p(\mathbb{T}^d_\theta)$, then $\widetilde{x} \in L_p(\mathcal{N}_\theta)$ and $\|\widetilde{x}\|_p = \|x\|_p$, that is, $x \mapsto \widetilde{x}$ is an isometric embedding from $L_p(\mathbb{T}^d_\theta)$ into $L_p(\mathcal{N}_\theta)$.*

iii) *Let $\widetilde{\mathbb{T}}^d_\theta = \{\widetilde{x} : x \in \mathbb{T}^d_\theta\}$. Then $\widetilde{\mathbb{T}}^d_\theta$ is a von Neumann subalgebra of $\mathcal{N}_\theta$ and the associated conditional expectation is given by*

$$\mathbb{E}(f)(z) = \pi_z\Big(\int_{\mathbb{T}^d} \pi_{\overline{w}}\big[f(w)\big]dw\Big), \quad z \in \mathbb{T}^d, \; f \in \mathcal{N}_\theta.$$

Moreover, $\mathbb{E}$ extends to a contractive projection from $L_p(\mathcal{N}_\theta)$ onto $L_p\left(\widetilde{\mathbb{T}}^d_\theta\right)$ for $1 \le p \le \infty$.

To avoid complicated notation, we will use the same notation for the derivation for the quantum tori $\mathbb{T}^d_\theta$ as for functions on $\mathbb{T}^d$. For every $1 \le j \le d$, define the derivation to be the operator ∂_j satisfying:

$$\partial_j(U_j) = 2\pi \mathrm{i} U_j \quad \text{and} \quad \partial_j(U_k) = 0 \text{ for } k \ne j.$$

Given $m \in \mathbb{N}^d_0$, the associated partial derivation D^m is $\partial_1^{m_1} \cdots \partial_d^{m_d}$. We keep using the resolvent of unit given by functions in (7.1). The Fourier multiplier on $\mathbb{T}^d_\theta$ with symbol $\varphi(2^{-j}\cdot)$ is then

$$\widetilde{\varphi}_j * x = \sum_{m \in \mathbb{Z}^d} \varphi(2^{-j}m)\, \widehat{x}(m) U^m.$$

The analogue of Schwartz class on the quantum torus is given by

$$\mathcal{S}(\mathbb{T}^d_\theta) = \{ \sum_{m \in \mathbb{Z}^d} a_m U^m : \{a_m\}_{m \in \mathbb{Z}^d} \text{ rapidly decreasing}\}.$$

This is a w^*-dense $*$-subalgebra of $\mathbb{T}^d_\theta$ and contains all polynomials. It is equipped with a structure of Fréchet $*$-algebra, and has a locally convex topology induced by a family of semi-norms. We denote the tempered distribution on $\mathbb{T}^d_\theta$ by $\mathcal{S}'(\mathbb{T}^d_\theta)$ which is the space of all continuous linear functional on $\mathcal{S}(\mathbb{T}^d_\theta)$. Then by duality, both partial derivations and the Fourier transform extend to $\mathcal{S}'(\mathbb{T}^d_\theta)$. Sobolev, Besov, and Triebel-Lizorkin spaces on the quantum torus are defined and well studied in [59]. Let us recall the definition.

Definition 7.9. *Let $1 \le p < \infty$ and $\alpha \in \mathbb{R}^d$. The potential Sobolev spaces are defined to be*

$$H^\alpha_p(\mathbb{T}^d_\theta) = \big\{x \in \mathcal{S}'(\mathbb{T}^d_\theta) : J^\alpha x \in L_p(\mathbb{T}^d_\theta)\big\},$$

equipped with the norm $\|x\|_{H^\alpha_p} = \|J^\alpha x\|_p$. Let also $1 \le q \le \infty$. The Besov spaces are defined by

$$B^\alpha_{p,q}(\mathbb{T}^d_\theta) = \big\{x \in \mathcal{S}'(\mathbb{T}^d_\theta) : \|x\|_{B^\alpha_{p,q}} < \infty\big\},$$

where

$$\|x\|_{B^\alpha_{p,q}} = \Big(|\widehat{x}(0)|^q + \sum_{k \ge 0} 2^{qk\alpha} \|\widetilde{\varphi}_k * x\|_p^q\Big)^{\frac{1}{q}}.$$

The column Triebel-Lizorkin spaces $F^{\alpha,c}_p(\mathbb{T}^d_\theta)$ are defined by

$$F^{\alpha,c}_p(\mathbb{T}^d_\theta) = \{x \in \mathcal{S}'(\mathbb{T}^d_\theta) : \|x\|_{F^{\alpha,c}_p} < \infty\},$$

where

$$\|x\|_{F^{\alpha,c}_p} = |\widehat{x}(0)| + \big\|(\sum_{j \ge 0} 2^{2j\alpha} |\widetilde{\varphi}_j * x|^2)^{\frac{1}{2}}\big\|_p.$$

The row space $F^{\alpha,r}_p(\mathbb{T}^d_\theta)$ and mixture space $F^\alpha_p(\mathbb{T}^d_\theta)$, and the case $p = \infty$ are defined similar to those on the usual d-torus.

The transference method in Lemma 7.8 allows us to connect the spaces defined above with their operator-valued counterparts. The result is

Lemma 7.10. *Let $1 \le p,q \le \infty$, and $\alpha \in \mathbb{R}$. The map $x \mapsto \widetilde{x}$ is an isometric embedding from $H_p^\alpha(\mathbb{T}_\theta^d)$, $B_{p,q}^\alpha(\mathbb{T}_\theta^d)$ and $F_p^{\alpha,c}(\mathbb{T}_\theta^d)$ into $H_p^\alpha(\mathbb{T}^d; L_p(\mathbb{T}_\theta^d))$, $B_{p,q}^\alpha(\mathbb{T}^d; L_p(\mathbb{T}_\theta^d))$ and $F_p^{\alpha,c}(\mathbb{T}^d, \mathbb{T}_\theta^d)$ respectively. Moreover, the ranges of these embeddings are 1-complemented in their respective spaces.*

Let us introduce toroidal symbol classes and pseudo-differential operators on $\mathbb{T}_\theta^d$. The following definitions were also given in [30].

Definition 7.11. *Let $0 \le \delta, \rho \le 1$, $n \in \mathbb{R}$ and $\gamma, \beta \in \mathbb{N}_0^d$ be multi-indices. Then the toroidal symbol class $S^n_{\mathbb{T}_\theta^d,\rho,\delta}(\mathbb{Z}^d)$ consists of those functions $\sigma : \mathbb{Z}^d \to \mathbb{T}_\theta^d$ which satisfy*

$$\|D^\beta(\Delta_m^\gamma \sigma(m))\| \le C_{\beta,\gamma}(1+|m|)^{n-\rho|\gamma|_1+\delta|\beta|_1}, \quad \forall\, m \in \mathbb{Z}^d \text{ and } \forall\, \gamma, \beta \in \mathbb{N}_0^d.$$

Definition 7.12. *Let $\sigma \in S^n_{\mathbb{T}_\theta^d,\rho,\delta}(\mathbb{Z}^d)$. For any $x \in \mathbb{T}_\theta^d$, we define the corresponding toroidal pseudo-differential operator on $\mathbb{T}_\theta^d$ as follows:*

$$T_\sigma^c x = \sum_{m\in\mathbb{Z}^d} \sigma(m)\widehat{x}(m)U^m.$$

Now we are ready to prove the mapping property of pseudo-differential operators on quantum torus.

Theorem 7.13. *Let $\sigma \in S^0_{\mathbb{T}_\theta^d,1,\delta}(\mathbb{Z}^d)$.*

i) *If $0 \le \delta < 1$ and $\alpha \in \mathbb{R}$, then T_σ^c is bounded on $H_2^\alpha(\mathbb{T}_\theta^d)$, $B_{p,q}^\alpha(\mathbb{T}_\theta^d)$ and $F_p^{\alpha,c}(\mathbb{T}_\theta^d)$ for $1 \le p,q \le \infty$.*
ii) *If $\delta = 1$ and $\alpha > 0$, then T_σ^c is bounded on $H_2^\alpha(\mathbb{T}_\theta^d)$ and $B_{p,q}^\alpha(\mathbb{T}_\theta^d)$ for $1 \le p,q \le \infty$, and bounded on $F_1^{\alpha,c}(\mathbb{T}_\theta^d)$.*
iii) *If $\delta = 1$, $\alpha \in \mathbb{R}$ and $(T_\sigma^c)^*$ admits a symbol in the class $S^0_{\mathbb{T}_\theta^d,1,1}(\mathbb{Z}^d)$, then T_σ^c is bounded on $F_p^{\alpha,c}(\mathbb{T}_\theta^d)$ for any $1 < p < \infty$.*

Proof. Recall that π_z denotes the isomorphism of $\mathbb{T}_\theta^d$ determined by (7.5). We claim that, given $m \in \mathbb{Z}^d$, the function $z \mapsto \pi_z(\sigma(m))$ from $\mathbb{T}^d$ to $\mathbb{T}_\theta^d$ satisfies

$$\|D_z^\gamma \Delta_m^\beta \pi_z(\sigma(m))\| \le C_{\gamma,\beta}(1+|m|)^{n+\delta|\gamma|_1-\rho|\beta|_1}. \tag{7.6}$$

Since π_z commutes with the derivations on $\mathbb{T}_\theta^d$, we have $D^\gamma\Delta^\beta\pi_z\sigma(m) = \pi_z(D^\gamma\Delta^\beta\sigma(m))$. Therefore,

$$\|D^\gamma\Delta^\beta\pi_z\sigma(m)\| = \|\pi_z(D^\gamma\Delta^\beta\sigma(m))\| \le \|D^\gamma\Delta^\beta\sigma(m)\| \le C_{\gamma,\beta}(1+|m|)^{n+\delta|\gamma|_1-\rho|\beta|_1}.$$

Denote $\widetilde{\sigma}(z,m) = \pi_z(\sigma(m))$ for $(z,m) \in \mathbb{T}^d \times \mathbb{Z}^d$ and consider the pseudo-differential operator $T^c_{\widetilde{\sigma}}$. Combining (7.6) and Theorem 7.7, we obtain the boundedness of $T^c_{\widetilde{\sigma}}$ on $F_p^{\alpha,c}(\mathbb{T}^d, \mathbb{T}_\theta^d)$. Moreover, for any polynomial x on $\mathbb{T}_\theta^d$ and $f(z) = \pi_z(x)$, we have

$$\begin{aligned} T^c_{\widetilde{\sigma}} f(z) &= \sum_{m\in\mathbb{Z}^d} \widetilde{\sigma}(z,m)\widehat{f}(m)z^m \\ &= \sum_{m\in\mathbb{Z}^d} \pi_z(\sigma(m))\widehat{x}(m)U^m z^m \\ &= \sum_{m\in\mathbb{Z}^d} \pi_z(\sigma(m)\widehat{x}(m)U^m) = \pi_z(T_\sigma^c(x)). \end{aligned}$$

Finally, by Lemma 7.10 and Theorem 7.7, we have

$$\begin{aligned} \|T_\sigma^c(x)\|_{F_p^{\alpha,c}(\mathbb{T}_\theta^d)} &= \|\pi_\cdot(T_\sigma^c(x))\|_{F_p^{\alpha,c}(\mathbb{T}^d,\mathbb{T}_\theta^d)} = \|T^c_{\widetilde{\sigma}} f\|_{F_p^{\alpha,c}(\mathbb{T}^d,\mathbb{T}_\theta^d)} \\ &\lesssim \|f\|_{F_p^{\alpha,c}(\mathbb{T}^d,\mathbb{T}_\theta^d)} = \|x\|_{F_p^{\alpha,c}(\mathbb{T}_\theta^d)}. \end{aligned}$$

The assertions on Sobolev and Besov spaces are proved similarly. □

Finally, let $0 \le \rho \le 1$, $n \in \mathbb{R}$ and $\gamma \in \mathbb{N}_0^d$. Define $S_\rho^n(\mathbb{Z}^d)$ as the scalar-valued toroidal symbol class, consisting of those functions $\sigma : \mathbb{Z}^d \to \mathbb{C}$ which satisfy

$$|\Delta_m^\gamma \sigma(m)| \le C_\gamma(1+|m|)^{n-\rho|\gamma|_1}, \quad \forall\, m \in \mathbb{Z}^d \text{ and } \forall\, \gamma \in \mathbb{N}_0^d.$$

In this setting, it is evident that T^c_σ and T^r_σ degenerate into the same Fourier multiplier on $\mathbb{T}^d_\theta$, simply denoted by T_σ. The following result is a Mikhlin-type Fourier multiplier theorem on $\mathbb{T}^d_\theta$.

Corollary 7.14. *Let $\sigma \in S^0_1(\mathbb{Z}^d)$ and $\alpha \in \mathbb{R}$. Then T_σ is bounded on $F^{\alpha,c}_p(\mathbb{T}^d_\theta)$, $F^{\alpha,r}_p(\mathbb{T}^d_\theta)$ and $F^{\alpha}_p(\mathbb{T}^d_\theta)$ for every $1 \le p \le \infty$.*

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