# KERNELS FOR PRODUCTS OF HILBERT L-FUNCTIONS

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Abstract. We study kernel functions of $L$-functions and products of $L$-functions of Hilbert cusp forms over real quadratic fields. This extends the results on elliptic modular forms in [4, 5].

## 1. Introduction

One of the central problems in number theory is to explore the nature of special values of various Dirichlet series such as Riemann zeta function, modular $L$-functions, automorphic $L$-functions, etc. The known main idea to study arithmetic properties of the special values of modular $L$-functions is to compare such values with certain inner product of modular forms.

Such an idea was first introduced by Rankin [13], expressing the product of two critical $L$-values of an elliptic Hecke eigenform in terms of the Petersson scalar product of an elliptic Hecke eigenform with a product of Eisenstein series. Much later Zagier ([16], p 149 ) extended Rankin's result to express the product of any two critical $L$-values of an elliptic Hecke eigenform in terms of the Petersson scalar product of the Hecke eigenform with the Rankin-Bracketof two Eisenstein series. Shimura [14] and Manin [11] developed theories to study arithmetic properties of modular $L$-values on the critical strip. Kohnen-Zagier [10, 2] further studied the space of modular forms whose $L$-values on the critical strip are rational and showed that such a space can be spanned by Cohen kernel introduced by Cohen [3]. Recently double Eisenstein series has been introduced by Diamantis and O'Sullivan[4, 5] as a kernel yielding products of two $L$-values of elliptic Hecke eigenforms. It turns out that Rankin-Cohen brackets [17] of two Eisenstein series can be realized as a double Eisenstein

Keynote: Hilbert modular form, special $L$-value, cusp form, double Eisenstein series, Petersson inner product, Rankin-Cohen bracket, kernel function.

2010 Mathematics Subject Classification: primary 11F67, 11F41 ; secondary 11F03 .

series [5]. Generalizing Cohen kernel, the arithmetic results of $L$-values by Manin [11] and Shimura [14] could be recovered [4, 5].

The purpose of this paper to state above results to the space of Hilbert modular forms by extending kernel functions introduced in [4, 5]. More precisely, a double Hilbert Eisenstein series is a kernel function of two $L$-values of a primitive form in terms of the Petersson scalar product. Also one can recover the arithmetic results [14] of $L$-values of Hilbert cusp forms by studying Cohen kernel over real quadratic fields. Furthermore it turns out that the Rankin-Cohen bracket of two Hilbert Eisenstein series is the special value of a double Hilbert Eisenstein series.

**Acknowledgment**: The first author was supported by NRF2018R1A4A1023 590 and NRF2017R1A2B2001807. The second author was partially supported by by HIT Youth Talent Start-Up Grant and Grant of Technology Division of Harbin (RC2016XK001001).

We thank the referee for the valuable remarks, which led to improvements in the paper.

## 2. Notations and Main Theorems

Throughout of this paper, for simplicity, we only consider the space of Hilbert modular forms over real quadratic fields $F$ with narrow class number one on the full Hilbert modular group $\Gamma = \mathrm{SL}_2(\mathcal{O})$.

2.1. **Notations.** Let $F$ be a real quadratic field with narrow class number equal to 1. Let $D$, $\mathcal{O}$ and $\mathfrak{d}$ be the fundamental discriminant, the ring of integers and the different of $F$ respectively. Let N and Tr be the norm and the trace on $F$ defined by $\mathrm{N}(a) = aa', \mathrm{Tr}(a) = a + a'$ with $a'$ the algebraic conjugate of $a \in F$. We denote $a \gg 0$ for $a \in F$ if $a$ is totally positive, that is $a > 0$ and $a' > 0$. For $B \subset F$, let $B_+$ denote the subset of totally positive elements in $B$. So $\mathcal{O}_+$ and $\mathcal{O}_+^\times$ denote the set of totally positive integers and the set of totally positive units respectively.

For a $2\times 2$ matrix $\gamma$ in $GL_2^+(F)$, we usually denote its entries by $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$ and $\gamma' = \begin{pmatrix} a'_\gamma & b'_\gamma \\ c'_\gamma & d'_\gamma \end{pmatrix}$. The group $GL_2^+(F)$ acts on two copies of the complex upper half plane $\mathbb{H}^2$ by $\gamma z := (\gamma z_1, \gamma' z_2) = (\frac{a_\gamma z_1+b_\gamma}{c_\gamma z_1+d_\gamma}, \frac{a'_\gamma z_2+b'_\gamma}{c'_\gamma z_2+d'_\gamma})$ as linear fractional transformations for all $\gamma \in GL_2^+(F)$ and $z=(z_1,z_2)\in\mathbb{H}^2$.

Let $\Gamma = \mathrm{SL}_2(\mathcal{O})$ be the modular group of $2\times 2$ matrices with determinant equal to one over $\mathcal{O}$. Denote $\Gamma_\infty$ the subgroup of upper-triangular elements and $\Gamma_\infty^+$ the subgroup of elements with totally positive diagonal entries in $\Gamma_\infty$. Let $A$ denote the subgroup of diagonal elements in $\Gamma_\infty^+$, so $A = \{\mathrm{diag}(\varepsilon,\varepsilon^{-1})\colon \varepsilon \in \mathcal{O}_+^\times\}$. Throughout the note, we employ the standard multi-index notation. In particular, for $\gamma\in \mathrm{GL}_2^+(F)$, $z=(z_1,z_2)\in\mathbb{H}^2$ and $k\in\mathbb{Z}$, we denote $\mathbf{1}=(1,1)$, $(\gamma z)^{k\mathbf{1}} = N(\gamma z)^k = (\gamma z_1)(\gamma' z_2)$, $|z| = (|z_1|,|z_2|)$, $|z|^{k\mathbf{1}} = |z_1|^k|z_2|^k$ and the automorphic factor by

$$j(\gamma,z)^{k\mathbf{1}} = N(j(\gamma,z))^k = j(\gamma,z_1)^k j(\gamma',z_2)^k = (c_\gamma z_1+d_\gamma)^k(c'_\gamma z_2+d'_\gamma)^k.$$

For any function $f$ on $\mathbb{H}^2$ and $\gamma\in\mathrm{GL}_2^+(F)$, define the slash operator by

$$(f|_k\gamma)(z) = N(\det(\gamma))^{\frac{k}{2}} N(j(\gamma,z))^{-k} f(\gamma z).$$

A Hilbert modular form of (parallel) weight $k$ for $\Gamma$ is a holomorphic function $f$ on $\mathbb{H}^2$ such that $f|_k\gamma = f$ for any $\gamma\in\Gamma$. Then $f$ has the following Fourier expansion

$$f(z) = a_f(0) + \sum_{\alpha\in\mathfrak{d}_+^{-1}} a_f(\alpha) e^{2\pi i \mathrm{tr}(\alpha z)}.$$

If $a_f(0)=0$, we call $f$ a Hilbert cusp form. For a Hilbert cusp form $f$ and a Hilbert modular form $g$ of weight $k$ on $\Gamma$, their Petersson scalar product is defined by

$$\langle f\,, g\rangle := \int_{\Gamma\backslash\mathbb{H}^2} f(z)\overline{g(z)}d\mu = \int_{\mathcal{F}} f(z)\overline{g(z)}d\mu,$$

where $\mathcal{F}$ is a fundamental domain of $\Gamma$ on $\mathbb{H}^2$ and

$$d\mu = (y_1y_2)^{-2}dx_1dx_2dy_1dy_2 = N(y)^{-2}N(dx)N(dy).$$

Here $z=x+iy$, $Re(z) = x = (x_1,x_2)$ and $Im(z) = y = (y_1,y_2)$.

Note that this "unnormalized" Petersson inner product is different from Shimura's [14]. For a Hilbert cusp form $f$ of weight $k$ for $\Gamma$, define the associated L-function by

$$L(f,s)=\sum_{\alpha\in\mathfrak{d}_+^{-1}/\mathcal{O}_+^\times}a_f(\alpha)N(\alpha\mathfrak{d})^{-s}=\sum_{\mathfrak{a}}a_f(\mathfrak{a})N(\mathfrak{a})^{-s},$$

where $a_f(\mathfrak{a}):=a_f(\alpha)$ for $\alpha\mathfrak{d}=\mathfrak{a}$. It is known [7] that the complete $L$-function satisfies

$$\Lambda(f,s):=D^s(2\pi)^{-2s}\Gamma(s)^2L(f,s)=(-1)^k\Lambda(f,k-s)$$

and has an analytic continuation to the entire $\mathbb{C}$.

Next we recall the theory of Hecke operators on spaces of Hilbert modular forms. For each nonzero integral ideal $\mathfrak{n}$ of $\mathcal{O}$, let $M_\mathfrak{n}$ be the set of $2\times 2$ matrices $\gamma$ over $\mathcal{O}$ such that $\det(\gamma)\gg 0$ and $(\det(\gamma))=\mathfrak{n}$. Moreover, let $Z\cong\mathcal{O}^\times$ denote the $2\times 2$ scalar matrices with diagonal entries in $\mathcal{O}^\times$. The $\mathfrak{n}$-th Hecke operator $T_\mathfrak{n}$ on $S_k(\Gamma)$, the space of cusp forms for $\Gamma$ of parallel weight-$k$, is defined as

$$T_\mathfrak{n}(f(z))=N(\mathfrak{n})^{\frac{k}{2}-1}\sum_{\gamma\in Z\Gamma\backslash M_\mathfrak{n}}f|_k\gamma(z).$$

The operators $T_\mathfrak{n}$ are self-adjoint with respect to the Petersson inner product and generate a commutative algebra. It follows that there exists a basis $\mathcal{H}_k$, consisting of normalized cuspidal Hecke eigenforms, of $S_k(\Gamma)$. We call elements in $\mathcal{H}_k$ "primitive forms". Here $f$ is normalized if the Fourier coefficient $a_f(\mathcal{O})=1$ or equivalently if $\mathfrak{d}^{-1}=(\alpha)$ with $\alpha\gg 0$, then $a_f(\alpha)=1$. Therefore, for $f\in\mathcal{H}_k$, $T_\mathfrak{n}f=a_f(\mathfrak{n})f$, so $a_f(\mathfrak{n})$ is real. For details see Section 1.15 of [7].

2.2. **Main Theorems.** Fix $k\in\mathbb{Z}$. We define the *Cohen kernel* $\mathcal{C}_k^{Hil}(z;s)$ on $\mathbb{H}^2\times\mathbb{C}$ by

$$\mathcal{C}_k^{Hil}(z;s)=\frac{1}{2}c_{k,s,D}^{-2}\sum_{\gamma\in A\backslash\Gamma}(\gamma z)^{-s\mathbf{1}}j(\gamma,z)^{-k\mathbf{1}}, \tag{2.1}$$

with

$$c_{k,s,D}=\frac{D^{\frac{k-1}{2}}2^{2-k}\pi\Gamma(k-1)}{e^{\frac{\pi is}{2}}\Gamma(s)\Gamma(k-s)}$$

and $A = \{\mathrm{diag}(\varepsilon, \varepsilon^{-1})\colon \varepsilon \in \mathcal{O}_+^\times\}$. Note that if $k$ is odd, this definition gives zero function.

**Theorem 2.1.** (**Cohen kernel**) *Let $k \geq 4$ be even. Then the following hold:*

*(1) $\mathcal{C}_k^{Hil}(z;s)$ converges absolutely and uniformly on all compact subsets in the region given by*

$$1 < Re(s) < k-1, \quad z \in \mathbb{H}^2.$$

*(2) For each $s \in \mathbb{C}$,*

$$\mathcal{C}_k^{Hil}(z;s) = \sum_{f \in \mathcal{H}_k} \frac{\Lambda(f, k-s)}{\langle f, f\rangle} f(z),$$

*where $\mathcal{H}_k$ is the set of primitive forms s of weight $k$ on $\Gamma$.*

*(3) $\mathcal{C}_k^{Hil}(z;s)$ can be analytically continued to the whole s-plane and for each $s \in \mathbb{C}$, $\mathcal{C}_k^{Hil}(z;s)$ is a cusp form for $\Gamma$ of weight $k$ in $z$.*

Next we define the *double Eisenstein series* as follows: for $s, w \in \mathbb{C}, z \in \mathbb{H}^2$ and even integer $k \geq 6$,

$$E_{s,k-s}^{Hil}(z;w) = \sum_{\gamma,\delta \in \Gamma_\infty^+ \backslash \Gamma, c_{\gamma\delta^{-1}} \gg 0} (c_{\gamma\delta^{-1}})^{(w-1)\mathbf{1}} \left(\frac{j(\gamma,z)}{j(\delta,z)}\right)^{-s\mathbf{1}} j(\delta,z)^{-k\mathbf{1}},$$

and a completed double Eisenstein series by

$$E_{s,k-s}^{*,Hil}(z;w) = 2\alpha_{k,s,w,D} \cdot E_{s,k-s}^{Hil}(z;w)$$

with

$$\begin{aligned}\alpha_{k,s,w,D} \ &:= \ D^{k-w}\zeta_F(1-w+s)\zeta_F(1-w+k-s) \\ &\quad \times \left(e^{\frac{is\pi}{2}}(2\pi)^{w-k-1}2^{k-2}\frac{\Gamma(s)\Gamma(k-s)\Gamma(k-w)}{\Gamma(k-1)}\right)^2.\end{aligned}$$

Then we have the following:

**Theorem 2.2.** (**double Eisenstein series**) *Let $k \geq 6$ be even.*

*(1)* $E^{Hil}_{s,k-s}(z;w)$ *converges absolutely and uniformly on compact subsets in the region* $\mathcal{R}$ *of points* $(z,(s,w))$ *in* $\mathbb{H}^2\times\mathbb{C}^2$ *subject to*

$$2<Re(s)<k-2,\ Re(w)<\min\{Re(s)-1,k-1-Re(s)\}.$$

*(2)* $E^{*,Hil}_{s,k-s}(z;w)$ *has an analytic continuation to all* $s,w\in\mathbb{C}$ *and is a Hilbert cusp form of weight* $k$ *on* $\Gamma$ *as a function in* $z$.

*(3)*

$$E^{*,Hil}_{s,k-s}(\cdot;w)=\sum_{f\in\mathcal{H}_k}\frac{\Lambda(f,s)\Lambda(f,w)}{\langle f,f\rangle}f,$$

*where* $\mathcal{H}_k$ *is the set of primitive forms of weight* $k$.

*(4) For* $f\in\mathcal{H}_k$, $\langle E^{*,Hil}_{s,k-s}(\cdot;w),\ f\rangle=\Lambda(f,s)\Lambda(f,w),$ *for all* $s,w\in\mathbb{C}$.

*(5)* $E^{*,Hil}_{s,k-s}(z;w)$ *satisfies functional equations:*

$$E^{*,Hil}_{s,k-s}(z;w)=E^{*,Hil}_{w,k-w}(z;s),\quad E^{*,Hil}_{k-s,s}(z;w)=E^{*,Hil}_{s,k-s}(z;w).$$

The following gives a relation between Rankin-Cohen brackets and a double Eisenstein series. Rankin-Cohen brackets on spaces of Hilbert modular forms have been studied in [1]. Let us recall the definition of Rankin-Cohen brackets: for each $j=1,2$, let $f_j:\mathbb{H}^2\to\mathbb{C}$ be holomorphic, $k_j\in\mathbb{N}$and $\ell=(\ell_1,\ell_2),\nu=(\nu_1,\nu_2)\in\mathbb{Z}^2_{\geq 0}$. Define the $\nu$-th Rankin-Cohen bracket

$$[f_1,f_2]^{Hil}_{\nu}=\sum_{0\leq\ell_j\leq\nu_j,\,j=1,2}(-1)^{\ell_1+\ell_2}\binom{k_1\mathbf{1}+\nu-\mathbf{1}}{\nu-\ell}\binom{k_2\mathbf{1}+\nu-\mathbf{1}}{\ell}f_1^{(\ell)}f_2^{(\nu-\ell)}.$$

Here $f^{(\ell)}(z)=(\frac{\partial^{\ell_1+\ell_2}}{\partial{z_1}^{\ell_1}\partial{z_2}^{\ell_2}}f)(z)$ and $\binom{k\mathbf{1}+\nu-\mathbf{1}}{\nu-\ell}=\binom{k+\nu_1-1}{\nu_1-\ell}\binom{k+\nu_2-1}{\nu_2-\ell}$.

In the following, we only need parallel $\nu$, that is $\nu_1=\nu_2$.

**Theorem 2.3.** *(***Rankin-Cohen brackets and a double Eisenstein series***) For* $\nu\in\mathbb{Z}_{\geq 0}$ *and* $k_j\in 2\mathbb{N}, j=1,2,$ *we have*

$$\left(\frac{\Gamma(k_1)\Gamma(\nu+1)}{\Gamma(k_1+\nu)}\right)^2[E_{k_1},E_{k_2}]^{Hil}_{(\nu,\nu)}=4\left(\frac{\Gamma(k_2+\nu)}{\Gamma(k_2)}\right)^2E^{Hil}_{k_1+\nu,k_2+\nu}(z;\nu+1),$$

*where* $E_k(z)$ *is the usual Hilbert Eisenstein series of weight* $k$ *on* $\Gamma$ *defined by*

$$E_k(z):=\sum_{\gamma\in\Gamma^+_\infty\backslash\Gamma}j(\gamma,z)^{-k\mathbf{1}}.$$

**Remark 2.4.** (1) Cohen kernel (see [3] and [10]) is an elliptic cusp form $R_n$ of weight $2k$ on $\mathrm{SL}_2(\mathbb{Z})$ characterized by, for each $0 \le n \le 2k-2$,

$$\langle f,\, R_n\rangle = n!(2\pi)^{-n-1}L(f, n+1), \text{ for all } f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z})).$$

Diamantis and O'Sullivan in [4] generalized Cohen kernel $\mathcal{C}_k^{ell}(\tau, s)$ to get

$$\langle f,\, \mathcal{C}_k^{ell}(\tau, s)\rangle = \Gamma(s)(2\pi)^s L(f, s), s \in \mathbb{C}.$$

(2) Double Eisenstein series was introduced and studied in [4, 5] as a kernel yielding products of the periods of an elliptic Hecke eigenform at critical values as well as producing products of $L$-functions for Maass cusp forms.

In the following theorem, we recover Shimura's result on the algebraicity of critical values of $L(f, s)$ (Theorem 4.3 of [14]). For a primitive form $f$ of even weight $k$, let $\mathbb{Q}(f)$ denote the number field generated by the Fourier coefficients of $f$ over $\mathbb{Q}$.

**Theorem 2.5.** *(***rationality***) Let $f$ be a primitive form of even weight $k \ge 6$ for $\Gamma$. Then there exist complex numbers $\omega_\pm(f)$ with $\langle f,\, f\rangle = \omega_+(f)\omega_-(f)$ such that for even $m$ and odd $\ell$ with $1 \le m, \ell \le k-1$,*

*(1)*

$$\frac{\Lambda(f, m)}{w_+(f)}, \frac{\Lambda(f, \ell)}{w_-(f)} \in \mathbb{Q}(f),$$

*(2) for each $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$,*

$$\left(\frac{\Lambda(f, m)}{\omega_+(f)}\right)^\sigma = \frac{\Lambda(f^\sigma, m)}{\omega_+(f^\sigma)}, \quad \left(\frac{\Lambda(f, \ell)}{\omega_-(f)}\right)^\sigma = \frac{\Lambda(f^\sigma, \ell)}{\omega_-(f^\sigma)}.$$

**Remark 2.6.** (1) The above theorm is an analogous result of that for elliptic modular forms proved in [10] (Theorem in page 202). We can also extend the rationality easily to arbitrary L-values as did in Theorem 8.3 of [5].

(2) The above theorem is a special case of Shimura theorem (Theorem 4.3 in [14]) by taking $n = 2$, $\psi = 1$, and $k_1 = k_2 = k$.

## 3. Proofs

We need the following multi variable Lipschitz summation formula.

**Lemma 3.1.** (**multivariable Lipschitz summation formula**) *Assume that $Im(s) > 2$. For $z \in \mathbb{H}^2$,*

$$\sum_{x\in\mathcal{O}}(z+x)^{-s\mathbf{1}} = \frac{(2\pi)^{2s}}{e^{\pi is}\Gamma(s)^2 D^{1/2}} \sum_{\xi\in\mathfrak{d}_+^{-1}} N(\xi)^{s-1}\exp(2\pi i\, Tr(\xi z)),$$

*Proof.* By the multi-index notation,

$$\sum_{x\in\mathcal{O}}(z+x)^{-s\mathbf{1}} = \sum_{x\in\mathcal{O}} N(z+x)^{-s} = \sum_{x\in\mathcal{O}}(z_1+x)^{-s}(z_2+x')^{-s}.$$

Following [9], define

$$f(x) = N(x)^{s-1}\exp(2\pi i \mathrm{Tr}(xz))$$

for $x = (x_1, x_2) \gg 0$ and 0 otherwise, so for $\mathrm{Im}(s) > 2$ and $z \in \mathbb{H}^2$, $f$ is clearly $L^1$ on the quadratic space $V = \mathbb{R}^2$ with the trace form. The computation of [9, Theorem 1] shows that the Fourier transform $\hat{f}(w)$ is given by

$$\hat{f}(w) = \frac{\Gamma(s)^2}{(-2\pi i)^{2s}}(z+w)^{-s\mathbf{1}}, \quad w \in \mathbb{R}^2.$$

It is clear that for $x \in \mathbb{R}^2$,

$$|f(x)| + |\hat{f}(x)| \ll (1+||x||)^{-2-\delta}$$

for any positive $\delta$, where $||\cdot||$ is the Euclidean norm. Therefore, we may apply the Poisson summation formula (see page 252 of [15]), and for a general lattice $M$ in $V$ with integral dual lattice $M^\vee$, the Poisson summation formula reads

$$\sum_{\alpha\in M} f(\alpha) = \sqrt{|M/M^\vee|}\sum_{\alpha\in M^\vee}\hat{f}(\alpha).$$

Now set $M = \mathfrak{d}^{-1}$, then $M^\vee = \mathcal{O}$, $|M/M^\vee| = D$ and the Lipschitz summation formula follows easily. □

Now we prove Theorem 2.1 about Cohen kernel.

*Proof of Theorem 2.1* : To show the convergence, we follow the treatment of Section 1.15 in [7]. Firstly, we prove the uniform absolute convergence on compact subsets, using the fact that $L^1$-convergence implies uniform convergence on compact subset for series of holomorphic functions (See Lemma on Page 52 of [7]). It suffices to treat the case for $z$ in a small neighborhood $U$ such that $\overline{U}$ is compact, $N(\mathrm{Im}z) > X^{-1}$ and $N(\mathrm{Im}\gamma z) < X$ for any $\gamma \in \Gamma$ and $z \in U$ for fixed big $X > 0$. Note that this essentially picks a Siegel set where $\overline{U}$ lives. In this case, we only have to prove that

$$\int_{\Gamma\backslash\mathbb{H}^2_X} \sum_{\gamma\in A\backslash\Gamma} |N(j(\gamma,z))|^{-k}|\gamma z|^{-\sigma\mathbf{1}} d\mu(z) < \infty$$

where $\sigma = \mathrm{Re}(s)$ and $\mathbb{H}^2_X$ is the subset of $z$ with $N(\mathrm{Im}z) < X$ in $\mathbb{H}^2$. Here we denote $|z| = (|z_1|, |z_2|)$ and employ the multi-index notation. The left-hand side is bounded by

$$\begin{aligned}
&\leq X^{\frac{k}{2}} \int_{\Gamma\backslash\mathbb{H}^2_X} \sum_{\gamma\in A\backslash\Gamma} (\mathrm{Im}\gamma z)^{\frac{k}{2}\mathbf{1}} |\gamma z|^{-\sigma\mathbf{1}} d\mu(z) \\
&\ll \sum_{\gamma\in A\backslash\Gamma} \int_{\Gamma\backslash\mathbb{H}^2_X} (\mathrm{Im}\gamma z)^{\frac{k}{2}\mathbf{1}} |\gamma z|^{-\sigma\mathbf{1}} d\mu(z) \\
&= \int_{A\backslash\mathbb{H}^2_X} (\mathrm{Im}z)^{\frac{k}{2}\mathbf{1}} |z|^{-\sigma\mathbf{1}} d\mu(z).
\end{aligned}$$

The space $A\backslash\mathbb{H}^2_X$ can be viewed as a subspace of

$$\{(z_1, z_2)\colon N(\mathrm{Im}z) < X, Y^{-1} \leq y_1/y_2 \leq Y\}$$

for some positive $Y$ ($Y$ can be chosen as the smallest totally positive unit bigger than 1). Moreover, that $N(\mathrm{Im}(-1/z)) < X$ implies $N(|z|^2) > X^{-1}N(\mathrm{Im}z)$. For $1 < r < \sigma < k-1$, the last quantity is equal to

$$\begin{aligned}
&\int_{A\backslash\mathbb{H}^2_X} (N(\mathrm{Im}z))^{\frac{k}{2}} |Nz|^{-r-(\sigma-r)} d\mu(z) \\
\ll &\int_{A\backslash\mathbb{H}^2_X} (N(\mathrm{Im}z))^{\frac{k-\sigma+r}{2}} |Nz|^{-r} d\mu(z) \\
\ll &\int_{y_1y_2<X, Y^{-1}<y_1/y_2<Y} (N(\mathrm{Im}z))^{\frac{k-\sigma+r}{2}} (N(\mathrm{Im}z))^{1-r} \frac{dy_1dy_2}{(y_1y_2)^2} \\
= &\int_{y_1y_2<X, Y^{-1}<y_1/y_2<Y} (N(\mathrm{Im}z))^{\frac{k-\sigma-r}{2}-1} dy_1dy_2 < \infty
\end{aligned}$$

where in the third line we applied Equation (5.8) of [4] for the integration on $x$. This is part (1).

For part (2), first note that the absolutely uniformly convergence implies that $\mathcal{C}_k^{Hil}(z;s)$ converges to a Hilbert modular form in the strip $2<\sigma<k-1$ since $\mathcal{C}_k^{Hil}(z;s)$ is $\Gamma$-invariant with a proper automorphic factor. Secondly, we write

$$2c_{k,s,D}^2\cdot\mathcal{C}_k^{Hil}(z;s)=\sum_{\alpha\in A\backslash\Gamma_\infty^+}\sum_{\gamma\in\Gamma_\infty^+\backslash\Gamma}j(\alpha\gamma,z)^{-k\mathbf{1}}(\alpha\gamma z)^{-s\mathbf{1}}$$
$$=\sum_{\gamma\in\Gamma_\infty^+\backslash\Gamma}j(\gamma,z)^{-k\mathbf{1}}\sum_{\alpha\in A\backslash\Gamma_\infty^+}(\alpha\gamma z)^{-s\mathbf{1}}=\sum_{\gamma\in\Gamma_\infty^+\backslash\Gamma}j(\gamma,z)^{-k\mathbf{1}}\sum_{x\in\mathcal{O}}(\gamma z+x)^{-s\mathbf{1}}.$$

Applying the Lipschitz summation formula in Lemma 3.1 with $2<\sigma<k-1$, we have

$$\begin{aligned}2c_{k,s,D}^2\cdot\mathcal{C}_k^{Hil}(z;s)&=\frac{(2\pi)^{2s}}{e^{\pi is}\Gamma(s)^2D^{1/2}}\sum_{\gamma\in\Gamma_\infty^+\backslash\Gamma}j(\gamma,z)^{-k\mathbf{1}}\sum_{\xi\in\mathfrak{d}_+^{-1}}(N\xi)^{s-1}\exp(2\pi i\mathrm{Tr}(\xi\gamma z))\\
&=\frac{(2\pi)^{2s}}{e^{\pi is}\Gamma(s)^2D^{1/2}}\sum_{\xi\in\mathfrak{d}_+^{-1}}(N\xi)^{s-1}\sum_{\gamma\in\Gamma_\infty^+\backslash\Gamma}j(\gamma,z)^{-k\mathbf{1}}\exp(2\pi i\mathrm{Tr}(\xi\gamma z))\\
&=\frac{(2\pi)^{2s}}{e^{\pi is}\Gamma(s)^2D^{1/2}}\sum_{\xi\in\mathfrak{d}_+^{-1}/(\mathcal{O}_+^\times)^2}(N\xi)^{s-1}\\
&\qquad\times\sum_{u\in\mathcal{O}_+^\times}\sum_{\gamma\in\Gamma_\infty^+\backslash\Gamma}j(\gamma,z)^{-k\mathbf{1}}\exp(2\pi i\mathrm{Tr}(u^2\xi\gamma z))\\
&=\frac{(2\pi)^{2s}}{e^{\pi is}\Gamma(s)^2D^{1/2}}\sum_{\xi\in\mathfrak{d}_+^{-1}/(\mathcal{O}_+^\times)^2}(N\xi)^{s-1}\sum_{\gamma\in U\backslash\Gamma}j(\gamma,z)^{-k\mathbf{1}}\exp(2\pi i\mathrm{Tr}(\xi\gamma z)),\end{aligned}$$

where $U$ is the subgroup of elements of the form $\begin{pmatrix}1&x\\0&1\end{pmatrix}$ in $\Gamma$. On the other hand, recall the $\xi$-th Poincaré series [7]

$$P_k(z;\xi)=\sum_{\gamma\in U\backslash\Gamma}j(\gamma,z)^{-k\mathbf{1}}\exp(2\pi i\mathrm{Tr}(\xi\gamma z))$$

and that it is a cusp form with

$$P_k(z;\xi)=\frac{\Gamma(k-1)^2D^{1/2}}{(4\pi)^{2k-2}N(\xi)^{k-1}}\sum_{f\in\mathcal{H}_k}\frac{\bar{a}_f(\xi)f}{\langle f,f\rangle}.$$

We see that up to a constant factor (depending on $s$) $\mathcal{C}_k^{Hil}(z;s)$ is equal to

$$\begin{aligned}
&\sum_{\xi\in\mathfrak{d}_+^{-1}/(\mathcal{O}_+^\times)^2}(N\xi)^{s-1}(N\xi)^{1-k}\sum_{f\in\mathcal{H}_k}\frac{\bar{a}_f(\xi)f(z)}{\langle f,f\rangle}\\
=&\sum_{\xi\in\mathfrak{d}_+^{-1}/(\mathcal{O}_+^\times)^2}(N\xi)^{s-k}\sum_{f\in\mathcal{H}_k}\frac{\bar{a}_f(\xi)f(z)}{\langle f,f\rangle}\\
=&2\sum_{\xi\in\mathfrak{d}_+^{-1}/\mathcal{O}_+^\times}(N\xi)^{s-k}\sum_{f\in\mathcal{H}_k}\frac{\bar{a}_f(\xi)f(z)}{\langle f,f\rangle}\\
=&2D^{k-s}\sum_{f\in\mathcal{H}_k}\frac{f(z)}{\langle f,f\rangle}\sum_{\xi\in\mathfrak{d}_+^{-1}/\mathcal{O}_+^\times}(N\xi\mathfrak{d})^{s-k}\bar{a}_f(\xi)\\
=&2D^{k-s}\sum_{f\in\mathcal{H}_k}\frac{f(z)L(f,k-s)}{\langle f,f\rangle},
\end{aligned}$$

where we used the fact that $a_f(\xi)$ is real. Putting everything together, we see that

$$2c_{k,s,D}^2\cdot\mathcal{C}_k^{Hil}(z;s)=\frac{2^{5-2k}\pi^2\Gamma(k-1)^2}{e^{\pi is}\Gamma(s)^2\Gamma(k-s)^2}\sum_{f\in\mathcal{H}_k}\frac{\Lambda(f,k-s)f(z)}{\langle f,f\rangle}$$

It follows that $\mathcal{C}_k^{Hil}(z;s)$ is cuspidal on the region $2<\sigma<k-1$, and that

$$\mathcal{C}_k^{Hil}(z;s)=\sum_{f\in\mathcal{H}_k}\frac{\Lambda(f,k-s)f(z)}{\langle f,f\rangle}.$$

For part (3): The expression of $\mathcal{C}_k^{Hil}(z;s)$ in part (2) gives the analytic continuation to $s\in\mathbb{C}$ and that for each $s\in\mathbb{C}$, $\mathcal{C}_k^{Hil}(z;s)$ is a cusp form. This completes the proof. □

Next, to prove Theorem 2.2 we first need to show a connection between Cohen kernel and double Eisenstein series, which is obtained in the following lemma:

**Lemma 3.2.** *On the region $\mathcal{R}$, we have*

$$\zeta_F(1-w+s)\zeta_F(1-w+k-s)E_{s,k-s}^{Hil}(z;w)=2c_{k,s,D}^2\sum_{\mathfrak{n}}N(\mathfrak{n})^{w-k}T_{\mathfrak{n}}\big(\mathcal{C}_k^{Hil}(z;s)\big),$$

*with $T_{\mathfrak{n}}$ the $\mathfrak{n}$-th Hecke operator and $\zeta_F(s)$ the Dedekind zeta function for $F$ defined as*

$$\zeta_F(s)=\sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}=\sum_{a\in\mathcal{O}_+/\mathcal{O}_+^\times} N(a)^{-s},$$

*where $\mathfrak{a}$ runs through all integral nonzero ideals.*

*Proof.* On $\mathcal{R}$, the series expansions of the two $\zeta_F$-factors converge absolutely. Therefore, on $\mathcal{R}$, by sending $\gamma$ to $(c_\gamma,d_\gamma)$, the left-hand side is equal to

$$\begin{aligned}&\zeta_F(1-w+s)\zeta_F(1-w+k-s)E^{Hil}_{s,k-s}(z;w)\\ =\ &\sum_{u,\tilde{u}} N(u)^{w-1-s}N(\tilde{u})^{w+s-1-k}\sum_{(c,d),(\tilde{c},\tilde{d})}(c\tilde{d}-d\tilde{c})^{(w-1)\mathbf{1}}\left(\frac{cz+d}{\tilde{c}z+\tilde{d}}\right)^{-s\mathbf{1}}(\tilde{c}z+\tilde{d})^{-k\mathbf{1}},\end{aligned}$$

where $u,\tilde{u}\in\mathcal{O}_+^\times\backslash\mathcal{O}_+$ and $(c,d),(\tilde{c},\tilde{d})\in\mathcal{O}_+^\times\backslash\mathcal{O}^2$ such that $\mathcal{O}c+\mathcal{O}d=\mathcal{O}\tilde{c}+\mathcal{O}\tilde{d}=\mathcal{O}$ and $c\tilde{d}-d\tilde{c}\gg 0$. Combining the two summations, we have

$$\sum_{\mathfrak{a},\tilde{\mathfrak{a}}}\sum_{(c,d),(\tilde{c},\tilde{d})}(c\tilde{d}-d\tilde{c})^{(w-1)\mathbf{1}}\left(\frac{cz+d}{\tilde{c}z+\tilde{d}}\right)^{-s\mathbf{1}}(\tilde{c}z+\tilde{d})^{-k\mathbf{1}},$$

where this time $\mathfrak{a},\tilde{\mathfrak{a}}$ are over all nonzero integral ideals and the inner summation is over $(c,d),(\tilde{c},\tilde{d})\in\mathcal{O}_+^\times\backslash\mathcal{O}^2$ such that $\mathcal{O}c+\mathcal{O}d=\mathfrak{a}$, $\mathcal{O}\tilde{c}+\mathcal{O}\tilde{d}=\tilde{\mathfrak{a}}$ and $c\tilde{d}-d\tilde{c}\gg 0$. Then we can remove the summation over $\mathfrak{a},\tilde{\mathfrak{a}}$ and it equals to

$$\begin{aligned}&\sum_{(c,d),(\tilde{c},\tilde{d})}(c\tilde{d}-d\tilde{c})^{(w-1)\mathbf{1}}\left(\frac{cz+d}{\tilde{c}z+\tilde{d}}\right)^{-s\mathbf{1}}(\tilde{c}z+\tilde{d})^{-k\mathbf{1}}\\ =&\sum_{\mathfrak{n}}\sum_{(c,d),(\tilde{c},\tilde{d})}(c\tilde{d}-d\tilde{c})^{(w-1)\mathbf{1}}\left(\frac{cz+d}{\tilde{c}z+\tilde{d}}\right)^{-s\mathbf{1}}(\tilde{c}z+\tilde{d})^{-k\mathbf{1}},\end{aligned}$$

where $\mathfrak{n}$ is over all nonzero integral ideals and the inner summation is over over $(c,d),(\tilde{c},\tilde{d})\in\mathcal{O}_+^\times\backslash\mathcal{O}^2$ such that $c\tilde{d}-d\tilde{c}\gg 0$ and $(c\tilde{d}-d\tilde{c})=\mathfrak{n}$. Note that the two summations over $(c,d),(\tilde{c},\tilde{d})$ in the preceding equation have different ranges.

Let $\tilde{A}$ denote the group of diagonal $2\times 2$ matrices with entries in $\mathcal{O}_+^\times$, so clearly $\tilde{A}\subset Z\Gamma$ and $\tilde{A}\backslash Z\Gamma\cong A\backslash\Gamma$. Note that the inner summation set is mapped bijectively to $\tilde{A}\backslash M_{\mathfrak{n}}$ via

$$((c,d),(\tilde{c},\tilde{d})) \mapsto \begin{pmatrix} c & d \\ \tilde{c} & \tilde{d} \end{pmatrix}.$$

Therefore, above expression is equal to

$$\begin{aligned}
&\sum_{\mathfrak{n}} \sum_{\gamma\in\tilde{A}\backslash M_{\mathfrak{n}}} (\det(\gamma))^{(w-1)\mathbf{1}}(\gamma z)^{-s} j(\gamma,z)^{-k\mathbf{1}} \\
=&\sum_{\mathfrak{n}} \sum_{\gamma\in Z\Gamma\backslash M_{\mathfrak{n}}} \sum_{\beta\in\tilde{A}\backslash Z\Gamma} (\det(\beta\gamma))^{(w-1)\mathbf{1}}(\beta\gamma z)^{-s\mathbf{1}} j(\beta\gamma,z)^{-k\mathbf{1}} \\
=&\sum_{\mathfrak{n}} \sum_{\gamma\in Z\Gamma\backslash M_{\mathfrak{n}}} \sum_{\beta\in A\backslash \Gamma} (\det(\gamma))^{(w-1)\mathbf{1}}(\beta\gamma z)^{-s\mathbf{1}} j(\beta\gamma,z)^{-k\mathbf{1}} \\
=&2c_{k,s,D}^2 \cdot \sum_{\mathfrak{n}} N(\mathfrak{n})^{-\frac{k}{2}+w-1} \sum_{\gamma\in Z\Gamma\backslash M_{\mathfrak{n}}} \mathcal{C}_k^{Hil}(z;s)|_k\gamma \\
=&2c_{k,s,D}^2 \cdot \sum_{\mathfrak{n}} N(\mathfrak{n})^{w-k} T_{\mathfrak{n}}\big(\mathcal{C}_k^{Hil}(z;s)\big),
\end{aligned}$$

which is the right-hand side. □

Using the preceding lemma we prove the following main theorem:

*Proof of Theorem 2.2* : For part (1), apply the proof of Lemma 4.1 in [5] for each component and we have

$$N(c_{\gamma\delta^{-1}}) \leq N(\mathrm{Im}(\gamma z))^{-1/2} N(\mathrm{Im}(\delta z))^{-1/2},$$

for any $\gamma,\delta\in\Gamma$ with $c_{\gamma\delta^{-1}} \gg 0$. Let $r=\max\{\mathrm{Re}(w),1\}$. Since $[\Gamma_\infty : \Gamma_\infty^+]$ is finite, $E_{s,k-s}(z;w)$ is absolutely bounded up to a constant by

$$\begin{aligned}
&\sum_{\gamma,\delta\in\Gamma_\infty^+\backslash\Gamma, c_{\gamma\delta^{-1}}\gg 0} (Nc_{\gamma\delta^{-1}})^{\mathrm{Re}(w)-1} |Nj(\gamma,z)|^{-\mathrm{Re}(s)} |Nj(\delta,z)|^{\mathrm{Re}(s)-k} \\
\leq &\sum_{\gamma,\delta\in\Gamma_\infty^+\backslash\Gamma, c_{\gamma\delta^{-1}}\gg 0} N(\mathrm{Im}(\gamma z))^{\frac{1-r}{2}} N(\mathrm{Im}(\delta z))^{\frac{1-r}{2}} |Nj(\gamma,z)|^{-\mathrm{Re}(s)} |Nj(\delta,z)|^{\mathrm{Re}(s)-k} \\
\leq &\sum_{\gamma,\delta\in\Gamma_\infty^+\backslash\Gamma, c_{\gamma\delta^{-1}}\neq 0} N(\mathrm{Im}(\gamma z))^{\frac{1-r}{2}} N(\mathrm{Im}(\delta z))^{\frac{1-r}{2}} |Nj(\gamma,z)|^{-\mathrm{Re}(s)} |Nj(\delta,z)|^{\mathrm{Re}(s)-k} \\
\ll &\sum_{\gamma,\delta\in\Gamma_\infty\backslash\Gamma, c_{\gamma\delta^{-1}}\neq 0} N(\mathrm{Im}(\gamma z))^{\frac{1-r}{2}} N(\mathrm{Im}(\delta z))^{\frac{1-r}{2}} |Nj(\gamma,z)|^{-\mathrm{Re}(s)} |Nj(\delta,z)|^{\mathrm{Re}(s)-k}
\end{aligned}$$

$$\ll N(y)^{-\frac{k}{2}} \sum_{\gamma,\delta\in\Gamma_\infty\backslash\Gamma, c_{\gamma\delta^{-1}}\neq 0} N(\mathrm{Im}(\gamma z))^{\frac{\mathrm{Re}(s)-r+1}{2}} N(\mathrm{Im}(\delta z))^{\frac{k-\mathrm{Re}(s)-r+1}{2}}$$
$$\ll N(y)^{-\frac{k}{2}} \sum_{\gamma,\delta\in\Gamma_\infty\backslash\Gamma} N(\mathrm{Im}(\gamma z))^{\frac{\mathrm{Re}(s)-r+1}{2}} N(\mathrm{Im}(\delta z))^{\frac{k-\mathrm{Re}(s)-r+1}{2}},$$

which is the product of two Eisenstein series whose absolute convergence is well-known (see, for example, 5.7 Lemma of Chapter I in [6]). So absolute convergence follows if we have

$$\frac{\mathrm{Re}(s)-r+1}{2} > 1 \quad \text{and} \quad \frac{k-\mathrm{Re}(s)-r+1}{2} > 1.$$

One sees easily that $E_{s,k-s}(z;w)$ transforms correctly under $\Gamma$. In the above estimate

$$\sum_{\gamma,\delta\in\Gamma_\infty\backslash\Gamma, c_{\gamma\delta^{-1}}\neq 0} N(\mathrm{Im}(\gamma z))^{\frac{\mathrm{Re}(s)-r+1}{2}} N(\mathrm{Im}(\delta z))^{\frac{k-\mathrm{Re}(s)-r+1}{2}}$$
$$= E\left(z, \frac{\mathrm{Re}(s)-r+1}{2}\right) E\left(z, \frac{k-\mathrm{Re}(s)-r+1}{2}\right) - E\left(z, \frac{k-2r+2}{2}\right),$$

where $E(z,s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} N(Im(\gamma z))^{-s} = N(y)^s + A(s)N(y)^{1-s} + o(1)$. By removing the highest terms $N(y)^{\frac{k}{2}-r+1}$ from the difference, the rest are all $o(N(y)^{\frac{k}{2}})$. This shows that $E_{s,k-s}(z;w) \to 0$ as $N(y)\to\infty$, and hence proves part (2) that $E_{s,k-s}(z;w)$ is a cuspform since only one cusp exists.

For part (3), by Theorem 2.1 the Cohen kernel are cuspforms. By Lemma 3.2,

$$\begin{aligned}
&\zeta_F(1-w+s)\zeta_F(1-w+k-s)E^{Hil}_{s,k-s}(z;w)\\
=&2c^2_{k,s,D}\sum_{\mathfrak{n}} N(\mathfrak{n})^{w-k} T_{\mathfrak{n}}\big(\mathcal{C}^{Hil}_k(z;s)\big)\\
=&2c^2_{k,s,D}\sum_{\mathfrak{n}} N(\mathfrak{n})^{w-k} \sum_{f\in\mathcal{H}_k} \frac{\langle T_{\mathfrak{n}}\mathcal{C}^{Hil}_k(z;s), f\rangle}{\langle f,f\rangle} f(z)\\
=&2c^2_{k,s,D}\sum_{\mathfrak{n}} N(\mathfrak{n})^{w-k} \sum_{f\in\mathcal{H}_k} \frac{\langle \mathcal{C}^{Hil}_k(z;s), T_{\mathfrak{n}}f\rangle}{\langle f,f\rangle} f(z)\\
=&2c^2_{k,s,D}\sum_{\mathfrak{n}} N(\mathfrak{n})^{w-k} \sum_{f\in\mathcal{H}_k} a_f(\mathfrak{n})\frac{\langle \mathcal{C}^{Hil}_k(z;s), f\rangle}{\langle f,f\rangle} f(z),
\end{aligned}$$

since $\overline{a_f(\mathfrak{n})} = a_f(\mathfrak{n})$. We have shown in Theorem 2.1 that

$$2c_{k,s,D}^2 \langle \mathcal{C}_k^{Hil}(z;s), f\rangle = \frac{2^{5-2k}\pi^2\Gamma(k-1)^2}{e^{\pi is}\Gamma(s)^2\Gamma(k-s)^2}\Lambda(f,k-s), \text{ for } f \in \mathcal{H}_k.$$

By defining

$$E_{s,k-s}^{*,Hil}(z;w) = 2\alpha_{k,s,w,D} E_{s,k-s}^{Hil}(z;w)$$

with $\alpha_{k,s,w,D}$ in (2.2) and using the result of Theorem 2.1, we obtain

$$E_{s,k-s}^{*,Hil}(z;w) = \sum_{f\in\mathcal{H}_k} \frac{\Lambda(f,k-w)\Lambda(f,k-s)}{\langle f,f\rangle} f(z) = \sum_{f\in\mathcal{H}_k} \frac{\Lambda(f,w)\Lambda(f,s)}{\langle f,f\rangle} f(z).$$

Part (4) follows easily from part (3) since $f$ is a primitive form. Finally, by part (3), $E_{s,k-s}^{*,Hil}(z;w)$ has meromorphic continuation to all of $s, w \in \mathbb{C}$, and reflected from properties of $\Lambda(f,s)$, it satisfies functional equations

$$E_{s,k-s}^{*,Hil}(z;w) = E_{w,k-w}^{*,Hil}(z;s), \quad E_{k-s,s}^{*,Hil}(z;w) = E_{s,k-s}^{*,Hil}(z;w),$$

proving part (5) and hence the whole theorem. □

Using the result about Rankin-Cohen brackets studied in [1], we prove Theorem 2.3:

*Proof of Theorem 2.3*: One checks (from Proposition 1 in [1])

$$\left(\frac{(k_1-1)!\nu!}{(k_1+\nu-1)!}\right)^2 [E_{k_1}, E_{k_2}]_{(\nu,\nu)}^{Hil} = \sum_{\delta\in\Gamma_\infty\backslash\Gamma} j(\delta,z)^{-k_1\mathbf{1}} E_{k_2}^{(\nu)}|_{k_2+2\nu}\delta.$$

Since

$$E_{k_2}^{(\nu)} = \left(\frac{(k_2-1+\nu)!}{(k_2-1)!}\right)^2 \sum_{\gamma\in\Gamma_\infty^+\backslash\Gamma} N(c_\gamma)^\nu j(\gamma,z)^{-(k_2+\nu)\mathbf{1}}$$

by Lemma 1 in [1], this in turn is equal to

$$\left(\frac{(k_2-1+\nu)!}{(k_2-1)!}\right)^2 \sum_{\delta,\gamma\in\Gamma_\infty^+\backslash\Gamma} j(\delta,z)^{-k_1\mathbf{1}} N(c_\gamma)^\nu j(\gamma,\delta(z))^{-(k_2+\nu)\mathbf{1}} j(\delta,z)^{-(k_2+2\nu)\mathbf{1}}$$

$$= \left(\frac{(k_2-1+\nu)!}{(k_2-1)!}\right)^2 \sum_{\delta,\gamma\in\Gamma_\infty^+\backslash\Gamma} N(c_{\gamma\delta^{-1}})^\nu j(\delta,z)^{-(k_1+\nu)\mathbf{1}} j(\gamma,z)^{-(k_2+\nu)\mathbf{1}}.$$

In such a particular situation, we see easily that the summand is actually well-defined on $\Gamma_\infty\backslash\Gamma$. Denote $S$ the subset of $(\delta,\gamma) \in (\Gamma_\infty^+\backslash\Gamma)^2$ with $c_{\gamma\delta^{-1}} \neq 0$ and

$S_{\pm,\pm} \subset S$ consists of elements whose $c_{\gamma\delta^{-1}}$ has the prescribed sign vector. In particular, $S_{+,+}$ consists of elements with $c_{\gamma\delta^{-1}} \gg 0$. It is obvious that the sums over these four subsets are all equal, since we may multiply on left by $\pm I$ and $\pm\mathrm{diag}(\varepsilon_0, \varepsilon_0^{-1}) \in \Gamma_\infty$ to adjust the signs; here $\varepsilon_0$ is the fundamental unit. That said, we have

$$\left(\frac{(k_1-1)!\nu!}{(k_1+\nu-1)!}\right)^2 [E_{k_1}, E_{k_2}]^{Hil}_{(\nu,\nu)} = 4\left(\frac{(k_2-1+\nu)!}{(k_2-1)!}\right)^2 E^{Hil}_{k_1+\nu,k_2+\nu}(z,\nu+1),$$

and it finishes the proof. □

*Proof of Theorem 2.5*: We follow the lines in Section 8A of [5] and first prove that for even $m$ and odd $\ell$ with $1 \le m, \ell \le k-1$, both of $E^{*,Hil}_{m,k-m}(z;k-1)$ and $E^{*,Hil}_{k-2,2}(z;\ell)$ have rational Fourier coefficients. By the functional equations in Theorem 2.2,

$$E^{*,Hil}_{m,k-m}(z;k-1) = E^{*,Hil}_{m,k-m}(z;1),$$

and it suffices to prove that the Fourier coefficients of $E^{*,Hil}_{m,k-m}(z;\ell)$ are rational for even $m$ and odd $\ell$ with $1 \le \ell < m \le k/2$. By Theorem 2.3, $E^{*,Hil}_{m,k-m}(z;\ell) = C[E_{m+1-\ell}, E_{k+1-m-\ell}]^{Hil}_{\ell-1}$, where $C$ is a rational multiple of $\pi^{2-2\ell}$ by Theorem 9.8 on page 515 of [12]. It follows that the Fourier coefficients of $E^{*,Hil}_{m,k-m}(z;\ell)$ belong to $\mathbb{Q}$.

Next, for primitive $f \in \mathcal{H}_k$, by Proposition 4.15 of [14] and Theorem 2.2, we have $\langle f, E^{*,Hil}_{k-1,2}(z;k-1)\rangle = \alpha_f\langle f, f\rangle = \Lambda(f,k-1)\Lambda(f,k-2)$, for certain $\alpha_f \in \mathbb{Q}(f)$. Again by Proposition 4.15, since $E^{*,Hil}_{k-1,2}(z;k-1)$ has rational Fourier coefficients, $\alpha_f^\sigma = \alpha_{f^\sigma}$ for each $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Also note $\alpha_f \ne 0$ because of the convergence of the Euler product of $\Lambda(f,s)$ for $\mathrm{Re}(s) \ge k/2+1$ (see Kim-Sarnak's bound in [8]). Define

$$\omega_+(f) = \frac{\alpha_f\langle f, f\rangle}{\Lambda(f,k-1)}, \quad \omega_-(f) = \frac{\langle f, f\rangle}{\Lambda(f,k-2)}.$$

Then for even $m$, odd $\ell$ with $1 \le m, \ell \le k-1$,

$$\frac{\Lambda(f,m)}{\omega_+(f)} = \frac{\langle f, E^{*,Hil}_{m,k-m}(z;k-1)\rangle}{\alpha_f\langle f, f\rangle} \in \mathbb{Q}(f)$$

again by Proposition 4.15 of [14] and similarly $\frac{\Lambda(f,\ell)}{\omega_-(f)} \in \mathbb{Q}(f)$. It is clear that $\omega_+(f)\omega_-(f) = \langle f, f\rangle$. Finally, the assertion (4.16) of [14] and that $\alpha_f^\sigma = \alpha_{f^\sigma}$

for each $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ implies that

$$\left(\frac{\Lambda(f,m)}{\omega_+(f)}\right)^\sigma = \frac{\Lambda(f^\sigma,m)}{\omega_+(f^\sigma)}, \quad \left(\frac{\Lambda(f,\ell)}{\omega_-(f)}\right)^\sigma = \frac{\Lambda(f^\sigma,\ell)}{\omega_-(f^\sigma)},$$

finishing the proof. □

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