L^1-DINI CONDITIONS AND LIMITING BEHAVIOR OF WEAK TYPE ESTIMATES FOR SINGULAR INTEGRALS

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Abstract. Let T_Ω be the singular integral operator with a homogeneous kernel Ω. In 2006, Janakiraman [10] showed that if Ω has mean value zero on $\mathbb{S}^{n-1}$ and satisfies the condition:

$$\sup_{|\xi|=1}\int_{\mathbb{S}^{n-1}}|\Omega(\theta)-\Omega(\theta+\delta\xi)|d\sigma(\theta)\le Cn\delta\int_{\mathbb{S}^{n-1}}|\Omega(\theta)|d\sigma(\theta),\quad (*)$$

where $0<\delta<\frac{1}{n}$, then the following limiting behavior

$$\lim_{\lambda\to 0_+}\lambda m(\{x\in\mathbb{R}^n:|T_\Omega f(x)|>\lambda\})=\frac{1}{n}\|\Omega\|_1\|f\|_1\quad (**)$$

holds for $f\in L^1(\mathbb{R}^n)$ and $f\ge 0$.

In the present paper, we prove that if replacing the condition $(*)$ by a more general condition, the L^1-Dini condition, then the limiting behavior $(**)$ still holds for the singular integral T_Ω. In particular, we give an example which satisfies the L^1-Dini condition, but does not satisfy $(*)$. Hence, we improve essentially the above result given in [10]. To prove our conclusion, we show that the L^1-Dini conditions defined respectively via rotation and translation in $\mathbb{R}^n$ are equivalent (see Theorem 2.5 below), which may have its own interest in the theory of the singular integrals. Moreover, similar limiting behavior for the fractional integral operator $T_{\Omega,\alpha}$ with a homogeneous kernel is also established in this paper.

1. Introduction

Suppose that the function Ω defined on $\mathbb{R}^n\setminus\{0\}$ satisfies the following conditions:

$$\Omega(\lambda x)=\Omega(x),\quad \text{for any } \lambda>0 \text{ and } x\in\mathbb{R}^n\setminus\{0\},\tag{1.1}$$

$$\int_{\mathbb{S}^{n-1}}\Omega(\theta)d\sigma(\theta)=0\tag{1.2}$$

and $\Omega\in L^1(\mathbb{S}^{n-1})$, where $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$ and $d\sigma$ is the area measure on $\mathbb{S}^{n-1}$. Define the singular integral T_Ω by

$$T_\Omega f(x)=\text{p.v.}\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^n}f(y)dy.$$

2010 *Mathematics Subject Classification.* 42B20.

Key words and phrases. Limiting behavior, weak type estimate, singular integral operator, L^1-Dini condition.

The work is supported by NSFC (No.11371057, 11471033), SRFDP (No.20130003110003) and the Fundamental Research Funds for the Central Universities (No.2014KJJCA10).

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It is well known that if Ω is odd and $\Omega \in L^1(\mathbb{S}^{n-1})$ (or Ω is even and $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$), T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ (see [2]), that is,

$$\|T_\Omega f\|_p \le C_p \|f\|_p. \tag{1.3}$$

For $p = 1$, Seeger [12] showed that if $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$,

$$m(\{x \in \mathbb{R}^n : |T_\Omega f(x)| > \lambda\}) \le C_1 \frac{\|f\|_1}{\lambda}. \tag{1.4}$$

If Ω is an odd function, the usual Calderón-Zygmund method of rotation gives some information on the constant in (1.3). In fact, $C_p = \frac{\pi}{2} H_p \|\Omega\|_1$ (see [8]), where H_p denotes the L^p norm of the Hilbert transform ($1 < p < \infty$).

In 2004, Janakiraman [9] proved that the constants C_p in (1.3) and C_1 in (1.4) are at worst $C \log n \|\Omega\|_1$ if Ω satisfies (1.1), (1.2) and the following *regularity condition* :

$$\sup_{|\xi|=1} \int_{\mathbb{S}^{n-1}} |\Omega(\theta) - \Omega(\theta + \delta\xi)| d\sigma(\theta) \le Cn\delta \int_{\mathbb{S}^{n-1}} |\Omega(\theta)| d\sigma(\theta), \quad 0 < \delta < \frac{1}{n}, \tag{1.5}$$

where C is a constant independent of the dimension. Later in 2006, Janakiraman [10] extended this result to the limiting case. Before stating Janakiraman's result, we give some notation. Let μ be a signed measure on $\mathbb{R}^n$, which is absolutely continuous with respect to Lebesgue measure and $|\mu|(\mathbb{R}^n) < \infty$. Here $|\mu|$ is the total variation of μ. Define

$$T_\Omega \mu(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} d\mu(y). \tag{1.6}$$

Theorem A ([10])**.** *Suppose* Ω *satisfies* (1.1)*,* (1.2) *and the regularity condition* (1.5)*. Then*

$$\lim_{\lambda \to 0_+} \lambda m(\{x \in \mathbb{R}^n : |T_\Omega \mu(x)| > \lambda\}) = \frac{1}{n} \|\Omega\|_1 |\mu(\mathbb{R}^n)|.$$

As a consequence of Theorem A, Janakiraman showed that

Corollary A ([10])**.** *Let* $f \in L^1(\mathbb{R}^n)$ *and* $f \ge 0$*. Suppose* Ω *satisfies* (1.1)*,* (1.2) *and* (1.5)*, then*

$$\lim_{\lambda \to 0_+} \lambda m(\{x \in \mathbb{R}^n : |T_\Omega f(x)| > \lambda\}) = \frac{1}{n} \|\Omega\|_1 \|f\|_1. \tag{1.7}$$

The limiting behavior in (1.7) is very interesting since it gives some information on the best constant for the weak type (1,1) estimate of the homogeneous singular integral operator T_Ω in some sense. However, note that the condition (1.5) seems to be strong compared with the *Hörmander condition* (see also [13]):

$$\sup_{y \ne 0} \int_{|x|>2|y|} |K(x-y) - K(x)| dx < \infty, \tag{1.8}$$

where K is the kernel of the Calderón-Zygmund singular integral operator. Hence, it is natural to ask whether (1.7) still holds if replacing (1.5) by the Hörmander condition (1.8). The purpose of this paper is to give an affirmative answer to this question in the case $K(x) = \Omega(x)|x|^{-n}$.

Before stating our results, we give the definition of the L^1-Dini condition.

Definition 1.1 (L^1-Dini condition)**.** Let Ω satisfy (1.1). We say that Ω satisfies the L^1-Dini condition if:

(i) $\Omega \in L^1(\mathbb{S}^{n-1})$;

(ii) $\int_0^1 \frac{\omega_1(\delta)}{\delta} d\delta < \infty$, where ω_1 denotes the L^1 integral modulus of continuity of Ω defined by

$$\omega_1(\delta) = \sup_{\|\rho\| \leq \delta} \int_{\mathbb{S}^{n-1}} |\Omega(\rho\theta) - \Omega(\theta)| d\sigma(\theta),$$

here ρ is a rotation on $\mathbb{R}^n$ and $\|\rho\| := \sup\{|\rho x' - x'| : x' \in \mathbb{S}^{n-1}\}$.

Let us recall two important facts given in [1] and [3], respectively.

Lemma A ([1])**.** *If Ω satisfies the L^1-Dini condition, then $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$ and $K(x) = \Omega(x)|x|^{-n}$ satisfies the Hörmander condition* (1.8).

Lemma B ([3])**.** *If $K(x) = \Omega(x)|x|^{-n}$ satisfies the Hörmander condition* (1.8)*, then $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$ and Ω satisfies the L^1-Dini condition.*

By Lemma A and Lemma B, one can see immediately that for the kernel $K(x) = \Omega(x)|x|^{-n}$ the Hörmander condition (1.8) is equivalent to the L^1-Dini condition.

In Section 2, we will prove that the regularity condition (1.5) is stronger than the L^1-Dini condition (see Proposition 2.1). Also we will give an example to show that the L^1-Dini condition is strictly weaker than the regularity condition (1.5) (see Example 2.2).

Our main goal in this paper is to prove that the limiting behavior (1.7) still holds if replacing the condition (1.5) by the L^1-Dini condition.

Theorem 1.2. *Suppose Ω satisfies* (1.1)*,* (1.2) *and the L^1-Dini condition. Let μ be an absolutely continuous signed measure on $\mathbb{R}^n$ with respect to Lebesgue measure and $|\mu|(\mathbb{R}^n) < \infty$. Define T_Ω by* (1.6)*. Then*

$$\lim_{\lambda \to 0_+} \lambda m(\{x \in \mathbb{R}^n : |T_\Omega \mu(x)| > \lambda\}) = \frac{1}{n} \|\Omega\|_1 |\mu(\mathbb{R}^n)|. \tag{1.9}$$

By setting $\mu(E) = \int_E f(x)dx$ with $f \in L^1(\mathbb{R}^n)$ in Theorem 1.2, we obtain the following result.

Corollary 1.3. *Let $f \in L^1(\mathbb{R}^n)$ and $f \geq 0$. Suppose Ω satisfies* (1.1)*,* (1.2) *and the L^1-Dini condition. Then*

$$\lim_{\lambda \to 0_+} \lambda m(\{x \in \mathbb{R}^n : |T_\Omega f(x)| > \lambda\}) = \frac{1}{n} \|\Omega\|_1 \|f\|_1.$$

The next results are related to the limiting behavior of the weak type estimate for the fractional integral operator $T_{\Omega,\alpha}$ with a homogenous kernel , which is defined as

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) dy, \quad 0 < \alpha < n.$$

The fractional integral operator $T_{\Omega,\alpha}$ which is a generalization of the Riesz potential, has been well studied (for example see the book [11] and the references therein). In [5], while studying the boundedness of $T_{\Omega,\alpha}$ on Hardy space, Ding and Lu introduced the following regularity condition for Ω:

$$\int_0^1 \frac{\omega_q(\delta)}{\delta^{1+\alpha}} d\delta < \infty, \tag{1.10}$$

where ω_q denotes the L^q integral modulus of continuity of Ω.

To study the limiting behavior of the fractional homogeneous operator, we need some regularity condition similar to (1.10). For convenience, we give the following notation.

Definition 1.4 (L^s_α-Dini condition)**.** Let Ω satisfy (1.1), $1 \le s \le \infty$ and $0 < \alpha < n$. We say that Ω satisfies the L^s_α-Dini condition if

(i) $\Omega \in L^s(\mathbb{S}^{n-1})$;

(ii) $\int_0^1 \frac{\omega_1(\delta)}{\delta^{1+\alpha}} d\delta < \infty$, where ω_1 is defined as that in Definition 1.1.

Let ν be an absolutely continuous signed measure on $\mathbb{R}^n$ with respect to Lebesgue measure and $|\nu|(\mathbb{R}^n) < \infty$. Define

$$T_{\Omega,\alpha}\nu(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} d\nu(y). \tag{1.11}$$

We have the following results for $T_{\Omega,\alpha}$, which is similar to T_Ω in Theorem 1.2.

Theorem 1.5. *Let ν be an absolutely continuous signed measure on $\mathbb{R}^n$ with respect to Lebesgue measure and $|\nu|(\mathbb{R}^n) < \infty$. Let $0 < \alpha < n$ and $r = \frac{n}{n-\alpha}$. Suppose Ω satisfies* (1.1)*,* (1.2) *and the L^r_α-Dini condition. Then*

$$\lim_{\lambda \to 0_+} \lambda^r m(\{x \in \mathbb{R}^n : |T_{\Omega,\alpha}\nu(x)| > \lambda\}) = \frac{1}{n}\|\Omega\|_r^r |\nu(\mathbb{R}^n)|^r.$$

Corollary 1.6. *Let $0 < \alpha < n$ and $r = \frac{n}{n-\alpha}$. Assume $f \in L^1(\mathbb{R}^n)$ and $f \ge 0$. Suppose Ω satisfies* (1.1)*,* (1.2) *and the L^r_α-Dini condition. Then*

$$\lim_{\lambda \to 0_+} \lambda^r m(\{x \in \mathbb{R}^n : |T_{\Omega,\alpha}f(x)| > \lambda\}) = \frac{1}{n}\|\Omega\|_r^r \|f\|_1^r.$$

We would like to point out the proof of Theorem 1.2 draws heavily on ideas from [10]. However, to establish the limiting behavior of the singular integral operator T_Ω with Ω satisfying the L^1-Dini condition, we need to carefully study the regularity of Ω. More precisely, we will show that two different L^1-Dini conditions are equivalent (see Theorem 2.5).

The paper is organized as follows. In Section 2, we give some properties of the L^1-Dini condition and the embedding relationship between the regularity condition (1.5) and the L^1-Dini condition. An example which shows the L^1-Dini condition is weaker than the condition (1.5) is also given in this section. The proof of Theorem 1.2 is given in Section 3. We outline the proof of Theorem 1.5 in the final section. Throughout this paper the letter C will stand for a positive constant which is not necessarily the same one in each occurrence.

2. L^1-Dini condition

In this section, we discuss some properties of the L^1-Dini condition. We first show that the regularity condition (1.5) is stronger than the L^1-Dini condition.

Proposition 2.1. *If Ω satisfies (1.1), (1.2) and the condition (1.5), then Ω satisfies the L^1-Dini condition.*

Proof. We first claim that if Ω satisfies (1.1), (1.2) and (1.5), then there exists $C > 0$ such that

$$\omega_1(\delta) \le \sup_{|\xi|=1} \int_{\mathbb{S}^{n-1}} |\Omega(\theta + C\delta\xi) - \Omega(\theta)| d\sigma(\theta) \tag{2.1}$$

holds for any $0 < \delta < \frac{1}{2}$. To prove (2.1), by Definition 1.1, it is enough to show that for any fixed $\theta \in \mathbb{S}^{n-1}$,

$$\{\rho\theta : \|\rho\| \le \delta\} \subset \Big\{ \frac{\theta + C\delta\xi}{|\theta + C\delta\xi|} : \ \xi \in \mathbb{S}^{n-1} \Big\}$$

for some constant $C > 0$. For convenience, set

$$A = \{\rho\theta : \|\rho\| \le \delta\}$$

and

$$B(C) = \Big\{ \frac{\theta + C\delta\xi}{|\theta + C\delta\xi|} : \ \xi \in \mathbb{S}^{n-1} \Big\}.$$

It is easy to see that $A = \{\eta \in \mathbb{S}^{n-1} : \ |\eta - \theta| \le \delta\}$. Choose $C = 2$. In the following, we will show that

$$B(2) \supset A. \tag{2.2}$$

Notice that the function $f(\xi) = \Big| \frac{\theta+2\delta\xi}{|\theta+2\delta\xi|} - \theta \Big|$ is continuous on $\mathbb{S}^{n-1}$. Since $\mathbb{S}^{n-1}$ is compact, then $f(\xi)$ can get its maximal value at a point of $\mathbb{S}^{n-1}$. Suppose ξ_0 is such a point that $f(\xi)$ achieves a maximum at ξ_0. Since $f(\theta) = 0$ and $f(-\theta) = 0$, ξ_0 must be located between θ and $-\theta$. Therefore again by the continuity of $f(\xi)$,

$$B(2) = \{\eta \in \mathbb{S}^{n-1} : \ |\eta - \theta| \le \gamma\} \quad \text{with} \quad \gamma = f(\xi_0).$$

So to prove (2.2), it suffices to show that $\gamma \ge \delta$. By rotation, we may suppose $\theta = (1, 0, 0, \cdots, 0)$. Choose $\xi = (0, 1, 0, \cdots, 0)$. Then

$$\gamma \ge \Big| \frac{\theta + 2\delta\xi}{|\theta + 2\delta\xi|} - \theta \Big| = \Big(2 - \frac{2}{\sqrt{1 + 4\delta^2}} \Big)^{\frac{1}{2}} \ge \delta.$$

Hence we prove (2.1) by choosing $C = 2$.

Now we split the integral $\int_0^1 \frac{\omega_1(\delta)}{\delta} d\delta$ into two parts:

$$\int_0^{\frac{1}{2n}} \frac{\omega_1(\delta)}{\delta} d\delta + \int_{\frac{1}{2n}}^1 \frac{\omega_1(\delta)}{\delta} d\delta.$$

For the first integral, using estimate (2.1) and the regularity condition (1.5), we can get the bound $C\|\Omega\|_1$. For the second integral, using $\omega_1(\delta) \le 2\|\Omega\|_1$ for any $0 < \delta < 1$, we can also get the bound $C\|\Omega\|_1$. Combining these, we conclude the proof. ∎

In the following, we give an example which satisfies (1.1), (1.2) and the L^1-Dini condition but does not satisfy the regularity condition (1.5).

Example 2.2. Consider the dimension $n = 2$. In this case, denote $\mathbb{S}^1 = \{\theta : \ 0 \le \theta \le 2\pi\}$, where θ is the arc length on the unit circle. Let $\Omega(\theta) = \theta^{-\frac{1}{2}} - (\frac{2}{\pi})^{\frac{1}{2}}$. It can be easily extended to the whole space $\mathbb{R}^2$ so that Ω is homogeneous of degree zero.

By using the parameter representation of arc length, the integral of Ω on $\mathbb{S}^1$ can be rewritten as

$$\int_0^{2\pi} \Omega(\theta)d\theta,$$

where θ is the arc length. Obviously, Ω in Example 2.2 satisfies (1.2).

Now let us first show that Ω in Example 2.2 does not satisfy the regularity condition (1.5). In fact, let δ be small enough. In $\mathbb{R}^2$, for any rotation $\|\rho\| \le \delta$, we have $\rho\theta = \theta \pm s$, where $s = \|\rho\|$. Consider the case $\rho\theta = \theta + s$, we get

$$\begin{aligned}\int_0^{2\pi} |\Omega(\rho\theta) - \Omega(\theta)| d\theta &= \int_0^{2\pi - s} \left| \frac{1}{\theta^{1/2}} - \frac{1}{(\theta + s)^{1/2}} \right| d\theta \\ &\quad + \int_{2\pi - s}^{2\pi} \left| \frac{1}{\theta^{1/2}} - \frac{1}{(\theta + s - 2\pi)^{1/2}} \right| d\theta \\ &= 4((2\pi - s)^{1/2} - (2\pi)^{1/2} + s^{1/2}) =: g(s),\end{aligned}$$

where in the first equality we use the fact that when $\theta \in (2\pi - s, 2\pi)$, $\rho\theta$ falls into $(0, s)$. A similar computation shows that if $\rho\theta = \theta - s$,

$$\int_0^{2\pi} |\Omega(\rho\theta) - \Omega(\theta)| d\theta = g(s).$$

It is not difficult to see that $g(s)$ is an increasing function for $s \in [0, \delta]$ and $g(0) = 0$. Therefore we have

$$\omega_1(\delta) = \sup_{\|\rho\| \le \delta} \int_0^{2\pi} |\Omega(\rho\theta) - \Omega(\theta)| d\theta = g(\delta).$$

Now by (2.1) in Lemma 2.1 (note that constant $C = 2$),

$$\begin{aligned}\frac{1}{2\delta} \sup_{|\xi|=1} \int_{\mathbb{S}^1} |\Omega(\theta + 2\delta\xi) - \Omega(\theta)| d\theta &\ge \frac{1}{2\delta}\omega_1(\delta) \\ &= 2\Big(\frac{1}{\delta^{1/2}} - \frac{(2\pi)^{1/2} - (2\pi - \delta)^{1/2}}{\delta}\Big) \to +\infty\end{aligned}$$

as $\delta \to 0$. This means that Ω does not satisfy the regularity condition (1.5). By a direct computation, we get

$$\int_0^1 \frac{\omega_1(\delta)}{\delta} d\delta = 4\int_0^1 \Big(\frac{1}{\delta^{1/2}} - \frac{(2\pi)^{1/2} - (2\pi - \delta)^{1/2}}{\delta}\Big) d\delta < \infty$$

and

$$\int_0^{2\pi} |\Omega(\theta)| d\theta < \infty,$$

which means that Ω satisfies the L^1-Dini condition in Definition 1.1.

In order to prove Theorem 1.2, we need to give an equivalent definition of the L^1-Dini condition in Definition 1.1.

Recall in Definition 1.1, the L^1-Dini condition is defined by the L^1 integral modulus ω_1, and ω_1 is defined by *rotation* in $\mathbb{R}^n$. In [1], Calderón, Weiss and Zygmund gave another L^1 integral modulus $\tilde{\omega}_1$ which is defined by *translation* in $\mathbb{R}^n$ as follows. Let Ω satisfy (1.1) and $\Omega \in L^1(\mathbb{S}^{n-1})$. Define $\tilde{\omega}_1$ as

$$\tilde{\omega}_1(\delta) = \sup_{|h|\le\delta} \int_{\mathbb{S}^{n-1}} |\Omega(x'+h) - \Omega(x')| d\sigma(x'), \tag{2.3}$$

where $h \in \mathbb{R}^n$. Similarly, one may define the L^1-Dini condition by the L^1 integral modulus $\tilde{\omega}_1$.

Definition 2.3. Let Ω satisfy (1.1). We say that Ω satisfies the L^1-Dini condition if:

(i) $\Omega \in L^1(\mathbb{S}^{n-1})$;

(ii) $\int_0^1 \frac{\tilde{\omega}_1(\delta)}{\delta} d\delta < \infty$, where $\tilde{\omega}_1(\delta)$ is defined by (2.3).

In [1], Calderón, Weiss and Zygmund pointed out that the L^1-Dini condition in Definition 1.1 is the most natural one. However, in some cases, the L^1-Dini definition in Definition 2.3 is more convenient in applications. Thus, a natural question to ask is whether there is a relationship between those two kinds of L^1-Dini conditions defined by Definition 1.1 and Definition 2.3.

Below we will show that these two kinds of L^1-Dini conditions defined by Definition 1.1 and Definition 2.3 are indeed equivalent. Let us first recall a useful lemma.

Lemma 2.4 (see Lemma 5 in [1])**.** *There exist positive constants α_0, C depending only on the dimension n such that if Ω is any function integrable over $\mathbb{S}^{n-1}$ and $0 < |h| \le \alpha_0$, $h \in \mathbb{R}^n$, then*

$$\int_{\mathbb{S}^{n-1}} |\Omega(\xi - h) - \Omega(\xi)| d\sigma(\xi) \le C \sup_{\|\rho\|\le|h|} \int_{\mathbb{S}^{n-1}} |\Omega(\rho\xi) - \Omega(\xi)| d\sigma(\xi). \tag{2.4}$$

Note that we may choose the constant α_0 in Lemma 2.4 less than 1.

Theorem 2.5. *L^1-Dini conditions defined respectively in Definition 1.1 and Definition 2.3 are equivalent.*

Proof. By Definition 1.1 and Definition 2.3, it is enough to show that for $\Omega \in L^1(\mathbb{S}^{n-1})$, the following conditions (a) and (b) are equivalent:

(a) $\int_0^1 \frac{\omega_1(\delta)}{\delta} d\sigma(\delta) < \infty$, where $\omega_1(\delta) = \sup\limits_{\|\rho\|\le\delta} \int_{\mathbb{S}^{n-1}} |\Omega(\rho x') - \Omega(x')| d\sigma(x')$,

(b) $\int_0^1 \frac{\tilde{\omega}_1(\delta)}{\delta} d\sigma(\delta) < \infty$, where $\tilde{\omega}_1(\delta) = \sup\limits_{|h|\le\delta} \int_{\mathbb{S}^{n-1}} |\Omega(x'+h) - \Omega(x')| d\sigma(x')$.

We first show that (b) implies (a). By (2.1) (note that the constant $C = 2$),

$$\omega_1(\delta) \le \sup_{|\xi|=1} \int_{\mathbb{S}^{n-1}} |\Omega(\theta + 2\delta\xi) - \Omega(\theta)| d\sigma(\theta) \le \tilde{\omega}_1(2\delta).$$

Hence we obtain

$$\begin{aligned}\int_0^1 \frac{\omega_1(\delta)}{\delta}d\delta &= \Big(\int_0^{1/2}+\int_{1/2}^1\Big)\frac{\omega_1(\delta)}{\delta}d\delta \le \int_0^{1/2}\frac{\tilde{\omega}_1(2\delta)}{\delta}d\delta+\int_{1/2}^1\frac{\omega_1(\delta)}{\delta}d\delta\\ &\le \int_0^1 \frac{\tilde{\omega}_1(\delta)}{\delta}d\delta + C\|\Omega\|_1.\end{aligned}$$

Now we turn to the other part: (a) implies (b). By Lemma 2.4, there exists a constant $0<a_0<1$ such that for any $0<|h|\le a_0$,

$$\int_{\mathbb{S}^{n-1}}|\Omega(\xi+h)-\Omega(\xi)|d\sigma(\xi)\le C\sup_{\|\rho\|\le|h|}\int_{\mathbb{S}^{n-1}}|\Omega(\rho\theta)-\Omega(\rho)|d\sigma(\theta).$$

If $0<\delta<a_0$, then

$$\begin{aligned}\tilde{\omega}_1(\delta)&=\sup_{|h|\le\delta}\int_{\mathbb{S}^{n-1}}|\Omega(\xi+h)-\Omega(\xi)|d\sigma(\xi)\\ &\le C\sup_{|h|\le\delta}\sup_{\|\rho\|\le|h|}\int_{\mathbb{S}^{n-1}}|\Omega(\rho\theta)-\Omega(\theta)|d\sigma(\theta)\le C\omega_1(\delta).\end{aligned}$$

If $a_0\le\delta<1$, we get

$$\tilde{\omega}_1(\delta)=\sup_{|h|\le\delta}\int_{\mathbb{S}^{n-1}}|\Omega(\theta+h)-\Omega(\theta)|d\sigma(\theta)\le\|\Omega\|_1+\sup_{|h|\le\delta}\int_{\mathbb{S}^{n-1}}|\Omega(\theta+h)|d\sigma(\theta).$$

Therefore if we can prove that

$$\sup_{|h|\le\delta}\int_{\mathbb{S}^{n-1}}|\Omega(\theta+h)|d\sigma(\theta)\le C\|\Omega\|_1, \tag{2.5}$$

then we conclude

$$\begin{aligned}\int_0^1\frac{\tilde{\omega}_1(\delta)}{\delta}d\delta&=\Big(\int_0^{a_0}+\int_{a_0}^1\Big)\frac{\tilde{\omega}_1(\delta)}{\delta}d\delta\le C\int_0^1\frac{\omega_1(\delta)}{\delta}d\delta+\int_{a_0}^1\frac{\tilde{\omega}_1(\delta)}{\delta}d\delta\\ &\le C\int_0^1\frac{\omega_1(\delta)}{\delta}d\delta+\int_{a_0}^1\frac{1}{\delta}\Big(\|\Omega\|_1+\sup_{|h|\le\delta}\int_{\mathbb{S}^{n-1}}|\Omega(\theta+h)|d\sigma(\theta)\Big)d\delta\\ &\le C\int_0^1\frac{\omega_1(\delta)}{\delta}d\delta+C\|\Omega\|_1.\end{aligned}$$

Hence, to complete the proof of Theorem 2.5, it remains to verify (2.5). By rotation, we may assume that $h=(h_1,0,\cdots,0)$, where $0<h_1<1$. Using the spherical coordinate formula on $\mathbb{S}^{n-1}$(see Appendix D in [7]), we can write

$$\begin{aligned}\int_{\mathbb{S}^{n-1}}\Big|\Omega\Big(\frac{x+h}{|x+h|}\Big)\Big|d\sigma(x)&=\int_{\varphi_1=0}^{\pi}\cdots\int_{\varphi_{n-2}=0}^{\pi}\int_{\varphi_{n-1}=0}^{2\pi}\Big|\Omega\Big(\frac{x(\varphi)+h}{|x(\varphi)+h|}\Big)\Big|\\ &\qquad\times|J(n,\varphi)|d\varphi_{n-1}\cdots d\varphi_1,\end{aligned}\tag{2.6}$$

where $x(\varphi)$ and $J(n,\varphi)$ are defined as

$$\begin{aligned}
x_1 &= \cos\varphi_1,\\
x_2 &= \sin\varphi_1\cos\varphi_2,\\
x_3 &= \sin\varphi_1\sin\varphi_2\cos\varphi_3,\\
&\dots\\
x_{n-1} &= \sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\cos\varphi_{n-1},\\
x_n &= \sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\sin\varphi_{n-1};\\
J(n,\varphi) &= (\sin\varphi_1)^{n-2}\cdots(\sin\varphi_{n-3})^2\sin\varphi_{n-2}.
\end{aligned}$$

Compared with $x(\varphi)$, $\frac{x(\varphi)+h}{|x(\varphi)+h|}$ can be written as $x(\theta)$ with $\theta_i=\varphi_i, 2\le i\le n-1$. This is most clearly understood from a geometric point of view, since $h=(h_1,0,\cdots,0)$. So we make a variable transform that maps $(\varphi_1,\varphi_2,\cdots,\varphi_{n-1})$ into $(\theta_1,\theta_2,\cdots,\theta_{n-1})$ such that

$$\begin{cases}
\frac{\cos\varphi_1+h_1}{\sqrt{1+2h_1\cos\varphi_1+h_1^2}} = \cos\theta_1, \ \frac{\sin\varphi_1}{\sqrt{1+2h_1\cos\varphi_1+h_1^2}} = \sin\theta_1,\\
\varphi_2 = \theta_2,\\
\dots\\
\varphi_{n-1} = \theta_{n-1}.
\end{cases}$$

Thus $\frac{x(\varphi)+h}{|x(\varphi)+h|}=x(\theta)$. It is easy to see

$$\tan\theta_1=\frac{\sin\varphi_1}{\cos\varphi_1+h_1}.$$

Then we have

$$d\theta_1=\Big(\arctan\frac{\sin\varphi_1}{\cos\varphi_1+h_1}\Big)'d\varphi_1=\frac{1+h_1\cos\varphi_1}{1+2h_1\cos\varphi_1+h_1^2}d\varphi_1.$$

Note that $0\le\varphi_1\le\pi$ and $0<h_1<1$, then $0<\theta_1<\pi$. Therefore the right side of (2.6) is bounded by

$$\begin{aligned}
&\int_{\theta_1=0}^{\pi}\cdots\int_{\theta_{n-2}=0}^{\pi}\int_{\theta_{n-1}=0}^{2\pi}|\Omega(x(\theta))||J(n,\theta)|\frac{(1+2\cos\varphi_1h_1+h_1^2)^{n/2}}{1+h_1\cos\varphi_1}d\theta_{n-1}\cdots d\theta_1\\
&\le 2^{n-1}\int_{\theta_1=0}^{\pi}\cdots\int_{\theta_{n-2}=0}^{\pi}\int_{\theta_{n-1}=0}^{2\pi}|\Omega(x(\theta))||J(n,\theta)|d\theta_{n-1}\cdots d\theta_1\\
&=2^{n-1}\int_{\mathbb{S}^{n-1}}|\Omega(x)|d\sigma(x),
\end{aligned}$$

where in the first inequality we use

$$\frac{1+2h_1\cos\varphi_1+h_1^2}{1+h_1\cos\varphi_1}\le 2$$

and $0<h_1<1$. Thus we finish the proof of (2.5). □

Remark 2.6. By Theorem 2.5, when applying the L^1-Dini condition, one may use its definition in Definition 1.1 or Definition 2.3 depending on the requirement of the application at hand.

The L^r_α-Dini condition that we introduce in Definition 1.4 is defined by rotation. It is natural to consider the translation version.

Definition 2.7. Let Ω satisfy (1.1), $1 \le s \le \infty$ and $0 < \alpha < n$. We say that Ω satisfies the L^s_α-Dini condition if

(i) $\Omega \in L^s(\mathbb{S}^{n-1})$;

(ii) $\int_0^1 \frac{\tilde{\omega}_1(\delta)}{\delta^{1+\alpha}} d\delta < \infty$, where $\tilde{\omega}_1$ is defined by (2.3).

By using the similar method as in the proof of Theorem 2.5, we obtain the following result.

Theorem 2.8. *Let $s \ge 1$ and $0 < \alpha < n$. L^s_α-Dini conditions defined respectively in Definition 1.4 and Definition 2.7 are equivalent.*

3. Proof of Theorem 1.2

In this section we give the proof of Theorem 1.2. Let us begin with some elementary facts.

3.1. Some elementary facts.

Lemma 3.1. *Let μ be a signed measure on $\mathbb{R}^n$. For $t > 0$, define $\mu_t(E) = \mu(\frac{E}{t})$. Suppose E is the μ_t measurable set. Then*

$$|\mu_t|(E) = |\mu|_t(E).$$

Proof. Since μ is a signed measure on $\mathbb{R}^n$, by the Hahn decomposition (see [6]), there exists a positive set P and a negative set N such that $P \bigcup N = \mathbb{R}^n$ and $P \bigcap N = \emptyset$. If P' and N' are another such pair, then $P \triangle P' (= N \triangle N')$ is null for the measure μ. Therefore $\mu^+(E) = \mu(E \cap P)$ and $\mu^-(E) = -\mu(E \cap N)$. Since the Hahn decomposition is unique, the pair tP and tN can be seen as the Hahn decomposition of μ_t. Then for any μ_t measurable set E,

$$\begin{aligned} |\mu_t|(E) &= (\mu_t)^+(E) + (\mu_t)^-(E) = \mu_t(E \cap tP) - \mu_t(E \cap tN) \\ &= \mu\Big(\frac{1}{t}E \cap P\Big) - \mu\Big(\frac{1}{t}E \cap N\Big) \\ &= |\mu|\Big(\frac{1}{t}E\Big) = |\mu|_t(E). \end{aligned}$$

□

Lemma 3.2. *Let μ be a nonnegative measure defined on $\mathbb{R}^n$ and $\mu(\mathbb{R}^n) = 1$. Suppose μ is absolutely continuous with respect to Lebesgue measure. Then for any $0 < \varepsilon < 1$, there exists a_ε, $0 < a_\varepsilon < \infty$, such that $\mu(B(0, a_\varepsilon)) = \varepsilon$.*

Proof. Since $\mu(\mathbb{R}^n) = 1$, there exists M, $0 < M < \infty$, such that $\mu(B(0, M)) \ge \varepsilon$.

Set $A_\varepsilon = \{r : \mu(B(0, r)) \ge \varepsilon\}$ and denote $a_\varepsilon = \inf\limits_{r \in A_\varepsilon} r$. It is easy to see that $a_\varepsilon \le M < \infty$. We claim that $\mu(B(0, a_\varepsilon)) = \varepsilon$. In fact, by the definition of infimum, for any $\alpha > 0$, there exists a $r \in A_\varepsilon$, which satisfies $a_\varepsilon < r < a_\varepsilon + \alpha$, such that $\mu(B(0, r)) \ge \varepsilon$. Hence

$$\mu(B(0, a_\varepsilon)) \ge \mu(B(0, r)) - \mu(B(0, r) \backslash B(0, a_\varepsilon)) \ge \varepsilon - \mu(B(0, a_\varepsilon + \alpha) \backslash B(0, a_\varepsilon)).$$

Note that $m\big(B(0,a_\varepsilon+\alpha)\backslash B(0,a_\varepsilon)\big)\to 0$ as $\alpha\to 0$. Since μ is absolutely continuous with respect to Lebesgue measure, $\mu(B(0,a_\varepsilon+\alpha)\backslash B(0,a_\varepsilon))\to 0$ as $\alpha\to 0$. So $\mu(B(0,a_\varepsilon))\geq\varepsilon$.

On the other hand, by the definition of a_ε, for any $0<r<a_\varepsilon$, we have $\mu(B(0,r))<\varepsilon$. Note that

$$\mu(B(0,a_\varepsilon))\leq\mu(B(0,r))+\mu(B(0,a_\varepsilon)\backslash B(0,r))<\varepsilon+\mu(B(0,a_\varepsilon)\backslash B(0,r)).$$

Since $\mu(B(0,a_\varepsilon)\backslash B(0,r))\to 0$ as $r\to a_\varepsilon$, then $\mu(B(0,a_\varepsilon))\leq\varepsilon$. Therefore the proof is complete. □

Lemma 3.3. *Let $0\leq\alpha<n$ and $r=\frac{n}{n-\alpha}$. For a fixed $\lambda>0$,*

$$\lambda^r m\Big(\Big\{x\in\mathbb{R}^n:\frac{|\Omega(x)|}{|x|^{n-\alpha}}>\lambda\Big\}\Big)=\frac{1}{n}\int_{\mathbb{S}^{n-1}}|\Omega(\theta)|^r d\sigma(\theta). \tag{3.1}$$

Proof. By changing to polar coordinates,

$$\begin{aligned} m\Big(\Big\{x\in\mathbb{R}^n:\frac{|\Omega(x)|}{|x|^{n-\alpha}}>\lambda\Big\}\Big) &= \int_{\mathbb{S}^{n-1}}\int_0^\infty \chi_{\{|\Omega(\theta)|/s^{n-\alpha}>\lambda\}}s^{n-1}dsd\sigma(\theta)\\ &= \int_{\mathbb{S}^{n-1}}\int_0^{(\frac{|\Omega(\theta)|}{\lambda})^{\frac{1}{n-\alpha}}} s^{n-1}dsd\sigma(\theta)\\ &= \frac{1}{n\cdot\lambda^r}\int_{\mathbb{S}^{n-1}}|\Omega(\theta)|^r d\sigma(\theta). \end{aligned}$$

□

Lemma 3.4. *Let μ be a absolutely continuous signed measure on $\mathbb{R}^n$ with respect to Lebesgue measure and $|\mu|(\mathbb{R}^n)<\infty$. Suppose Ω satisfies (1.1), (1.2) and the L^1-Dini condition. For any $\lambda>0$,*

$$\lambda m(\{x\in\mathbb{R}^n:|T_\Omega\mu(x)|>\lambda\})\leq C|\mu|(\mathbb{R}^n) \tag{3.2}$$

where the constant C only depends on Ω and the dimension.

Proof. Since μ is a absolutely continuous signed measure on $\mathbb{R}^n$ with respect to Lebesgue measure and $|\mu|(\mathbb{R}^n)<\infty$, by Radon-Nikodym's theorem (see [6]), there exists an integrable function f such that $d\mu(x)=f(x)dx$. Therefore we have

$$T_\Omega\mu(x)=T_\Omega f(x).$$

Now the rest of the proof can be found in the book [7]. By carefully examining the proof there, the weak (1,1) bound in (3.2) is $C(\|\Omega\|_1+\int_0^1\frac{\omega_1(s)}{s}ds)$. □

3.2. **A key lemma.** Now we give a lemma which plays a key role in the proof of Theorem 1.2.

Lemma 3.5. *Let μ be an absolutely continuous signed measure with respect to Lebesgue measure on $\mathbb{R}^n$ and $|\mu|(\mathbb{R}^n)<+\infty$. Suppose Ω satisfies (1.1), (1.2) and the L^1-Dini condition. Define T_Ω by (1.6). Then for any $\lambda>0$,*

$$\lim_{t\to 0_+}\lambda m(\{x\in\mathbb{R}^n:|T_\Omega\mu_t(x)|>\lambda\})=\frac{1}{n}\|\Omega\|_1|\mu(\mathbb{R}^n)|.$$

Proof. Without loss of generality, we may assume $|\mu|(\mathbb{R}^n) = 1$. Let δ be small enough such that $0 < \delta \ll 1$. For any fixed $\lambda > 0$, choose ε such that $0 < \varepsilon \leq \frac{1}{2}\delta\lambda$. By Lemma 3.2, there exists an a_ε with $0 < a_\varepsilon < \infty$, such that $|\mu|(B(0, a_\varepsilon)) = 1 - \varepsilon$. Set $\varepsilon_t = a_\varepsilon \cdot t$, by Lemma 3.1 we have

$$|\mu_t|(B(0, \varepsilon_t)) = |\mu|_t(B(0, \varepsilon_t)) = 1 - \varepsilon.$$

Let $\eta > \varepsilon_t$. For $x \in B(0, \eta)^c$ and $y \in B(0, \varepsilon_t)$, we can choose the minimal positive constant τ which satisfies

$$\frac{1-\tau}{|x|^n} \leq \frac{1}{|x-y|^n} \leq \frac{1+\tau}{|x|^n}. \tag{3.3}$$

Then $\tau \to 0_+$ as $t \to 0_+$.

Define $d\mu_t^1(x) = \chi_{B(0,\varepsilon_t)}(x) d\mu_t(x)$ and $d\mu_t^2(x) = \chi_{B(0,\varepsilon_t)^c}(x) d\mu_t(x)$, where χ_E is the characteristic function of E. By the linearity of T_Ω,

$$|T_\Omega \mu_t^1(x)| - |T_\Omega \mu_t^2(x)| \leq |T_\Omega \mu_t(x)| \leq |T_\Omega \mu_t^1(x)| + |T_\Omega \mu_t^2(x)|.$$

For any given $\lambda > 0$, define

$$F_\lambda^t = \{x \in \mathbb{R}^n : |T_\Omega \mu_t(x)| > \lambda\},$$

$$F_{1,\lambda}^t = \{x \in \mathbb{R}^n : |T_\Omega \mu_t^1(x)| > \lambda\}$$

and

$$F_{2,\lambda}^t = \{x \in \mathbb{R}^n : |T_\Omega \mu_t^2(x)| > \lambda\}.$$

Since Ω satisfies the L^1-Dini condition, by Lemma 3.4, T_Ω is of weak type (1,1). Therefore

$$\begin{aligned} m(F_{2,\delta\lambda}^t) &= m(\{x \in \mathbb{R}^n : |T_\Omega \mu_t^2(x)| > \delta\lambda\}) \leq \frac{C}{\delta\lambda}|\mu_t^2|(\mathbb{R}^n) \\ &= \frac{C}{\delta\lambda}|\mu_t|(B(0, \varepsilon_t)^c) \leq \frac{C\varepsilon}{\delta\lambda}. \end{aligned} \tag{3.4}$$

Since $F_{1,(1+\delta)\lambda}^t \subset F_{2,\delta\lambda}^t \cup F_\lambda^t$ and $F_\lambda^t \subset F_{2,\delta\lambda}^t \cup F_{1,(1-\delta)\lambda}^t$, by (3.4) we have the following estimate

$$-\frac{C\varepsilon}{\delta\lambda} + m(F_{1,(1+\delta)\lambda}^t) \leq m(F_\lambda^t) \leq \frac{C\varepsilon}{\delta\lambda} + m(F_{1,(1-\delta)\lambda}^t). \tag{3.5}$$

By the choice of ε and δ, $m(F_{1,(1+\delta)\lambda}^t)$ and $m(F_{1,(1-\delta)\lambda}^t)$ converges to $m(F_\lambda^t)$ as $t \to 0_+$ by (3.5). It is easy to see that

$$m(F_{1,(1+\delta)\lambda}^t) - \omega_n \eta^n \leq m(F_{1,(1+\delta)\lambda}^t \cap B(0, \eta)^c) \leq m(F_{1,(1+\delta)\lambda}^t)$$

where ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. We conclude that $m(F_{1,(1+\delta)\lambda}^t \cap B(0, \eta)^c)$ converges to $m(F_{1,(1+\delta)\lambda}^t)$ as $\eta \to 0_+$. Similarly, $m(F_{1,(1-\delta)\lambda}^t \cap B(0, \eta)^c)$ converges to $m(F_{1,(1-\delta)\lambda}^t)$ as $\eta \to 0_+$.

Now we split $T_\Omega \mu_t^1(x)$ into two parts:

$$T_\Omega \mu_t^1(x) = \lim_{\varepsilon' \to 0_+} \int_{|x-y|>\varepsilon'} \frac{\Omega(x)}{|x|^n} d\mu_t^1(y) + \lim_{\varepsilon' \to 0_+} \int_{|x-y|>\varepsilon'} \Big(\frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n}\Big) d\mu_t^1(y).$$

Using the triangle inequality, we obtain

$$\begin{aligned}
&\Big|\int_{|x-y|>\varepsilon'}\frac{\Omega(x)}{|x|^n}d\mu_t^1(y)\Big|-\int_{|x-y|>\varepsilon'}\Big|\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x)}{|x|^n}\Big|d|\mu_t^1|(y)\\
&\le\Big|\int_{|x-y|>\varepsilon'}\frac{\Omega(x-y)}{|x-y|^n}d\mu_t^1(y)\Big|\\
&\le\Big|\int_{|x-y|>\varepsilon'}\frac{\Omega(x)}{|x|^n}d\mu_t^1(y)\Big|+\int_{|x-y|>\varepsilon'}\Big|\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x)}{|x|^n}\Big|d|\mu_t^1|(y).
\end{aligned}\tag{3.6}$$

Denote

$$G_t:=\Big\{x\in B(0,\eta)^c:\lim_{\varepsilon'\to0_+}\int_{|x-y|>\varepsilon'}\Big|\frac{\Omega(x)}{|x|^n}-\frac{\Omega(x-y)}{|x-y|^n}\Big|d|\mu_t^1|(y)\ge2\delta\lambda\Big\}.$$

Since

$$\Big|\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x)}{|x|^n}\Big|\le\frac{|\Omega(x-y)-\Omega(x)|}{|x-y|^n}+|\Omega(x)|\Big|\frac{1}{|x-y|^n}-\frac{1}{|x|^n}\Big|,$$

we get $G_t\subset G_{t,1}\cap G_{t,2}$, where

$$G_{t,1}:=\Big\{x\in B(0,\eta)^c:\lim_{\varepsilon'\to0_+}\int_{|x-y|>\varepsilon'}\frac{|\Omega(x-y)-\Omega(x)|}{|x-y|^n}d|\mu_t^1|(y)\ge\delta\lambda\Big\}$$

and

$$G_{t,2}:=\Big\{x\in B(0,\eta)^c:\lim_{\varepsilon'\to0_+}\int_{|x-y|>\varepsilon'}|\Omega(x)|\Big|\frac{1}{|x-y|^n}-\frac{1}{|x|^n}\Big|d|\mu_t^1|(y)\ge\delta\lambda\Big\}.$$

First consider $G_{t,1}$. If $x\in B(0,\eta)^c$ and $y\in B(0,\varepsilon_t)$, then $|x|>|y|$ and $\frac{1}{|x-y|^n}\le\frac{1+\tau}{|x|^n}$ by (3.3). Using Chebyshev's inequality, Fubini's theorem and changing to polar coordinates, we have

$$\begin{aligned}
m(G_{t,1})&\le m\Big(\Big\{x\in B(0,\eta)^c:\int_{\mathbb{R}^n}\frac{|\Omega(x)-\Omega(x-y)|}{|x|^n}d|\mu_t^1|(y)\ge\frac{\delta\lambda}{1+\tau}\Big\}\Big)\\
&\le\frac{1+\tau}{\lambda\delta}\int_{B(0,\eta)^c}\int_{\mathbb{R}^n}\frac{|\Omega(x-y)-\Omega(x)|}{|x|^n}d|\mu_t^1|(y)dx\\
&=\frac{1+\tau}{\lambda\delta}\int_{\mathbb{R}^n}\int_{B(0,\eta)^c}\frac{|\Omega(x-y)-\Omega(x)|}{|x|^n}dxd|\mu_t^1|(y)\\
&=\frac{1+\tau}{\lambda\delta}\int_{\mathbb{R}^n}\int_\eta^{+\infty}\int_{\mathbb{S}^{n-1}}\Big|\Omega(\theta-\frac{y}{r})-\Omega(\theta)\Big|d\sigma(\theta)\cdot\frac{dr}{r}d|\mu_t^1|(y).
\end{aligned}$$

By Theorem 2.5, the L^1-Dini condition in Definition 2.3 and Definition 1.1 are equivalent. So in the following we use the L^1-Dini condition from Definition 2.3. Set $A(r):=\int_0^r\frac{\tilde{\omega}_1(s)}{s}ds$. Since Ω satisfies the L^1-Dini condition, we have $A(r)\to0$ as $r\to0_+$. Therefore

$$\begin{aligned}
m(G_{t,1})&\le\frac{(1+\tau)}{\lambda\delta}\int_{\mathbb{R}^n}\int_\eta^{+\infty}\frac{\tilde{\omega}_1(|y|/r)}{r}drd|\mu_t^1|(y)\\
&=\frac{(1+\tau)}{\lambda\delta}\int_{\mathbb{R}^n}\int_0^{|y|/\eta}\frac{\tilde{\omega}_1(s)}{s}dsd|\mu_t^1|(y)\\
&\le\frac{(1+\tau)}{\delta\lambda}\int_0^{\varepsilon_t/\eta}\frac{\tilde{\omega}_1(s)}{s}ds\int_{\mathbb{R}^n}d|\mu_t^1|(y)\\
&\le\frac{(1+\tau)}{\delta\lambda}A(\varepsilon_t/\eta),
\end{aligned}\tag{3.7}$$

where in the second equality we make the change of variable $|y|/r = s$.

Estimation of $m(G_{t,2})$ is similar to that of $m(G_{t,1})$. Again by using Chebyshev's inequality, Fubini's theorem, (3.3) and changing to polar coordinates,

$$\begin{aligned} m(G_{t,2}) &\le \frac{1}{\delta\lambda}\int_{B(0,\eta)^c}\int_{\mathbb{R}^n}|\Omega(x)|\Big|\frac{1}{|x|^n}-\frac{1}{|x-y|^n}\Big|d|\mu_t^1|(y)dx \\ &\le \frac{1}{\delta\lambda}\int_{\mathbb{R}^n}\int_{B(0,\eta)^c}|\Omega(x)|\frac{(1+\tau)n|y|}{|x|^{n+1}}dxd|\mu_t^1|(y) \\ &\le \frac{(1+\tau)n}{\delta\lambda}\|\Omega\|_1\int_{\mathbb{R}^n}\int_\eta^\infty\frac{dr}{r^2}|y|d|\mu_t^1|(y) \\ &\le \frac{(1+\tau)n\varepsilon_t}{\delta\lambda\eta}\|\Omega\|_1|\mu_t^1|(\mathbb{R}^n) \\ &\le \frac{(1+\tau)n\varepsilon_t}{\delta\lambda\eta}\|\Omega\|_1, \end{aligned} \tag{3.8}$$

where in the fourth inequality we use the fact $d\mu_t^1 = \chi_{B(0,\varepsilon_t)}d\mu_t$. Combining these estimates for $G_{t,1}$ and $G_{t,2}$, we get

$$m(G_t) \le m(G_{t,1}) + m(G_{t,2}) \le \frac{(1+\tau)}{\delta\lambda}A(\varepsilon_t/\eta) + \frac{(1+\tau)n\varepsilon_t}{\delta\lambda\eta}\|\Omega\|_1. \tag{3.9}$$

It is easy to see that

$$\begin{aligned} m(\{x\in B(0,\eta)^c\cap G_t^c : &|T_\Omega\mu_t^1(x)|>\lambda\}) \le m(\{F_{1,\lambda}^t\cap B(0,\eta)^c\}) \\ &\le m(\{x\in B(0,\eta)^c\cap G_t^c : |T_\Omega\mu_t^1(x)|>\lambda\}) + m(G_t). \end{aligned}$$

So if $x\in B(0,\eta)^c\cap G_t^c$, by the definition of G_t and (3.6),

$$\frac{|\Omega(x)|}{|x|^n}|\mu_t^1(\mathbb{R}^n)| - 2\delta\lambda \le |T_\Omega\mu_t^1(x)| \le \frac{|\Omega(x)|}{|x|^n}|\mu_t^1(\mathbb{R}^n)| + 2\delta\lambda.$$

Therefore we obtain

$$\begin{aligned} &\Big\{x\in B(0,\eta)^c\cap G_t^c : |T_\Omega\mu_t^1(x)|>(1-\delta)\lambda\Big\} \\ &\quad\subset \Big\{x\in B(0,\eta)^c\cap G_t^c : \frac{|\Omega(x)|}{|x|^n}|\mu_t^1(\mathbb{R}^n)|>(1-3\delta)\lambda\Big\} \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &\Big\{x\in B(0,\eta)^c\cap G_t^c :\ |T_\Omega\mu_t^1(x)|>(1+\delta)\lambda\Big\} \\ &\quad\supset \Big\{x\in B(0,\eta)^c\cap G_t^c : \frac{|\Omega(x)|}{|x|^n}|\mu_t^1(\mathbb{R}^n)|>(1+3\delta)\lambda\Big\}. \end{aligned} \tag{3.11}$$

By the definition of μ_t^1,

$$|\mu_t^1(\mathbb{R}^n)| = |\mu(\mathbb{R}^n) - \mu_t(B(0,\varepsilon_t)^c)|.$$

Note that $|\mu_t(B(0,\varepsilon_t)^c)| \le |\mu_t|(B(0,\varepsilon_t)^c) \le \varepsilon$, so we have

$$|\mu(\mathbb{R}^n)| - \varepsilon < |\mu_t^1(\mathbb{R}^n)| \le |\mu(\mathbb{R}^n)| + \varepsilon.$$

Using (3.9), (3.10), (3.11) and Lemma 3.3 with $\alpha = 0$,

$$
\begin{aligned}
&m(F^t_{1,(1+\delta)\lambda}) \\
&\geq m(\{x \in B(0,\eta)^c \cap G^c_t : |T_\Omega \mu^1_t(x)| > (1+\delta)\lambda\}) \\
&\geq m\Big(\Big\{x \in B(0,\eta)^c \cap G^c_t : \frac{|\Omega(x)|}{|x|^n}|\mu^1_t(\mathbb{R}^n)| \geq (1+3\delta)\lambda\Big\}\Big) \\
&\geq m\Big(\Big\{x \in \mathbb{R}^n : \frac{|\Omega(x)|}{|x|^n}|\mu^1_t(\mathbb{R}^n)| > (1+3\delta)\lambda\Big\}\Big) - \omega_n\eta^n - m(G_t) \\
&\geq \frac{\|\Omega\|_1}{n}\cdot\frac{|\mu(\mathbb{R}^n)| - \varepsilon}{(1+3\delta)\lambda} - \omega_n\eta^n - \frac{(1+\tau)}{\delta\lambda}A(\frac{\varepsilon_t}{\eta}) - \frac{(1+\tau)n\varepsilon_t}{\delta\lambda\eta}\|\Omega\|_1
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
&m(F^t_{1,(1-\delta)\lambda}) \\
&\leq m(\{x \in B(0,\tau)^c \cap G^c_t : |T_\Omega \mu^1_t(x)| > (1-\delta)\lambda\}) + m(B(0,\eta)) + m(G_t) \\
&\leq m\Big(\Big\{x \in \mathbb{R}^n : \frac{|\Omega(x)|}{|x|^n}|\mu^1_t(\mathbb{R}^n)| > (1-3\delta)\lambda\Big\}\Big) + \omega_n\eta^n + m(G_t) \\
&\leq \frac{\|\Omega\|_1}{n}\cdot\frac{|\mu(\mathbb{R}^n)| + \varepsilon}{(1-3\delta)\lambda} + \omega_n\eta^n + \frac{(1+\tau)}{\delta\lambda}A(\frac{\varepsilon_t}{\eta}) + \frac{(1+\tau)n\varepsilon_t}{\delta\lambda\eta}\|\Omega\|_1.
\end{aligned}
\tag{3.13}
$$

Here ω_n is the volume of the unit ball in $\mathbb{R}^n$. Combining the above estimates (3.12), (3.13) and (3.4), we conclude

$$
\begin{aligned}
m(F^t_\lambda) &\geq m(F^t_{1,(1+\delta)\lambda}) - m(F^t_{2,\delta\lambda}) \\
&\geq \frac{\|\Omega\|_1}{n}\frac{|\mu(\mathbb{R}^n)| - \varepsilon}{(1+3\delta)\lambda} - \omega_n\eta^n - \frac{(1+\tau)}{\delta\lambda}A(\frac{\varepsilon_t}{\eta}) - \frac{(1+\tau)n\varepsilon_t}{\delta\lambda\eta}\|\Omega\|_1 - \frac{C\varepsilon}{\delta\lambda}
\end{aligned}
$$

and

$$
\begin{aligned}
m(F^t_\lambda) &\leq m(F^t_{1,(1-\delta)\lambda}) + m(F^t_{2,\delta\lambda}) \\
&\leq \frac{\|\Omega\|_1}{n}\frac{|\mu(\mathbb{R}^n)| + \varepsilon}{(1-3\delta)\lambda} + \omega_n\eta^n + \frac{(1+\tau)}{\delta\lambda}A(\frac{\varepsilon_t}{\eta}) + \frac{(1+\tau)n\varepsilon_t}{\delta\lambda\eta}\|\Omega\|_1 + \frac{C\varepsilon}{\delta\lambda}.
\end{aligned}
$$

Let $t \to 0_+$, then $\varepsilon_t \to 0_+$ and $\tau \to 0_+$. So $A(\frac{\varepsilon_t}{\eta}) \to 0_+$. Then we obtain

$$\liminf_{t\to 0_+} m(F^t_\lambda) \geq \frac{\|\Omega\|_1}{n}\frac{|\mu(\mathbb{R}^n)| - \varepsilon}{(1+3\delta)\lambda} - \omega_n\eta^n - \frac{C\varepsilon}{\delta\lambda}$$

and

$$\limsup_{t\to 0_+} m(F^t_\lambda) \leq \frac{\|\Omega\|_1}{n}\frac{|\mu(\mathbb{R}^n)| + \varepsilon}{(1-3\delta)\lambda} + \omega_n\eta^n + \frac{C\varepsilon}{\delta\lambda}.$$

Note that $\varepsilon \leq \frac{1}{2}\delta\lambda$. Now let $\varepsilon \to 0_+$ first and $\delta \to 0_+$ second. Lastly let $\eta \to 0_+$. Then

$$\frac{\|\Omega\|_1|\mu(\mathbb{R}^n)|}{n\lambda} \leq \liminf_{t\to 0_+} m(F^t_\lambda) \leq \limsup_{t\to 0_+} m(F^t_\lambda) \leq \frac{\|\Omega\|_1|\mu(\mathbb{R}^n)|}{n\lambda},$$

which completes the proof. □

3.3. **Proof of Theorem 1.2.** We write $T_\Omega\mu_t(x)$ as

$$\begin{aligned}\lim_{\epsilon\to 0_+}\int_{|x-y|>\epsilon}\frac{\Omega(x-y)}{|x-y|^n}d\mu_t(y) &= \frac{1}{t^n}\lim_{\epsilon\to 0_+}\int_{|\frac{x-y}{t}|>\epsilon}\frac{\Omega\big(\frac{x}{t}-\frac{y}{t}\big)}{|\frac{x}{t}-\frac{y}{t}|^n}d\mu\big(\frac{y}{t}\big)\\ &= \frac{1}{t^n}T_\Omega\mu\big(\frac{x}{t}\big).\end{aligned} \tag{3.14}$$

Then by (3.14),

$$\begin{aligned}m(\{x\in\mathbb{R}^n : |T_\Omega\mu_t(x)|>\lambda\}) &= m\Big(\Big\{x\in\mathbb{R}^n : \frac{1}{t^n}|T_\Omega\mu\big(\frac{x}{t}\big)|>\lambda\Big\}\Big)\\ &= t^n m(\{x\in\mathbb{R}^n : |T_\Omega\mu(x)|>\lambda t^n\}).\end{aligned}$$

Applying Lemma 3.5, we get

$$\begin{aligned}\lim_{\lambda\to 0_+}\lambda m(\{x\in\mathbb{R}^n : |T_\Omega\mu(x)|>\lambda\}) &= \lim_{t\to 0_+}\lambda t^n m(\{x\in\mathbb{R}^n : |T_\Omega\mu(x)|>\lambda t^n\})\\ &= \lim_{t\to 0_+}\lambda m(\{x\in\mathbb{R}^n : |T_\Omega\mu_t(x)|>\lambda\})\\ &= \frac{1}{n}\|\Omega\|_1|\mu(\mathbb{R}^n)|.\end{aligned}$$

Hence we complete the proof of Theorem 1.2. □

4. Proof of Theorem 1.5

In this section, we give the proof of Theorem 1.5. The proof is quite similar to that of Theorem 1.2. So we shall be brief and only indicate necessary modifications here. We first set up a result for $T_{\Omega,\alpha}$ which is similar to Lemma 3.5.

Lemma 4.1. *Set $0<\alpha<n$ and $r=\frac{n}{n-\alpha}$. Let μ be an absolutely continuous signed measure with respect to Lebesgue measure on $\mathbb{R}^n$ and $|\mu|(\mathbb{R}^n)<+\infty$. Suppose Ω satisfies* (1.1), (1.2) *and the L^r_α-Dini condition. Then for any $\lambda>0$,*

$$\lim_{t\to 0_+}\lambda^r m(\{x\in\mathbb{R}^n : |T_{\Omega,\alpha}\mu_t(x)|>\lambda\}) = \frac{1}{n}\|\Omega\|_r^r|\mu(\mathbb{R}^n)|^r.$$

Proof. The proof is similar to that of Lemma 3.5. Choose the same constants δ, ε, a_ε and ε_t as we do in the proof of Lemma 3.5. For the constant τ we choose the minimal constant such that

$$\frac{1-\tau}{|x|^{n-\alpha}}\le\frac{1}{|x-y|^{n-\alpha}}\le\frac{1+\tau}{|x|^{n-\alpha}}.$$

Since $T_{\Omega,\alpha}$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)$ (see page 224 in [4]), we can get an estimate analogous to (3.4). For the estimate similar to $m(G_{t,1})$, by Theorem 2.8, we use the equivalent L^r_α-Dini condition in Definition 2.7. In the estimates similar to (3.12) and (3.13), we can use Lemma 3.3 with $0<\alpha<n$. Proceeding the proof as we do in the proof of Lemma 3.5, we may obtain the result of Lemma 4.1. □

Proof of Theorem 1.5. As we have done in the last part of section 3, we could establish the following dilation property of $T_{\Omega,\alpha}$ which is similar to (3.14):

$$T_{\Omega,\alpha}\mu_t(x) = \frac{1}{t^{n-\alpha}}T_{\Omega,\alpha}\mu\big(\frac{x}{t}\big).$$

By using the above equality and Lemma 4.1, we conclude

$$\begin{aligned}
&\lim_{\lambda\to 0_+}\lambda^r m(\{x\in\mathbb{R}^n : |T_{\Omega,\alpha}\mu(x)|>\lambda\})\\
&=\lim_{t\to 0_+}(\lambda t^{n-\alpha})^r m(\{x\in\mathbb{R}^n : |T_{\Omega,\alpha}\mu(x)|>\lambda t^{n-\alpha}\})\\
&=\lim_{t\to 0_+}\lambda^r m\Big(\Big\{x\in\mathbb{R}^n : |T_{\Omega,\alpha}\mu(\frac{x}{t})|>\lambda t^{n-\alpha}\Big\}\Big)\\
&=\lim_{t\to 0_+}\lambda^r m(\{x\in\mathbb{R}^n : |T_{\Omega,\alpha}\mu_t(x)|>\lambda\})=\frac{1}{n}\|\Omega\|_r^r|\mu(\mathbb{R}^n)|^r,
\end{aligned}$$

which completes the proof of Theorem 1.5. □

Acknowledgment. The authors would like to express their deep gratitude to the referee for his/her very careful reading, important comments and valuable suggestions.

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