

HALF-INTEGRAL WEIGHT MODULAR FORMS AND MODULAR FORMS FOR WEIL REPRESENTATIONS

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ABSTRACT. We establish an isomorphism between certain complex-valued and vector-valued modular form spaces of half-integral weight, generalizing the well-known isomorphism between modular forms for $\Gamma_0(4)$ with Kohnen's plus condition and modular forms for the Weil representation associated to the lattice with Gram matrix (2). With such an isomorphism, we prove the Zagier duality and express the Borcherds lifts in the case of $O(2, 1)$ explicitly.

INTRODUCTION

The theory of modular forms is of fundamental importance in many parts of modern number theory and many other related fields. Weakly holomorphic modular forms, namely those with possible poles at cusps, received less attention historically than the holomorphic ones. One of a few exceptions is the modular j -function, whose Fourier coefficients possess deep geometric, representation-theoretic and arithmetic information. The inverse of the Dedekind eta function, $\eta(\tau)^{-1}$, is another exception because of its direct connection with the partition function. Things changed when Borcherds, in his seminal papers [1] and [2], constructed a multiplicative theta lifting, also known as Borcherds automorphic product, which sends weakly holomorphic vector-valued modular forms of full level to modular forms in the form of infinite products for orthogonal groups. The Borcherds lift is in general a meromorphic modular form in the form of an infinite product, and Borcherds' theory shows precisely the location of its divisors.

Remarkably, in his work on traces of singular moduli, Zagier [21] proved a duality for Fourier coefficients of modular forms weights k and $2 - k$ for six small half-integral k , with which he gave a different proof of Borcherds's theorem for $\Gamma_0(4)$. Such duality is now known as Zagier duality and Zagier dualities for various types of modular forms, of integral or half-integral weight, have been proved since then (see [25] for a list of reference on this research).

Many important works have been built on Borcherds lifts by directly employing vector-valued modular forms ever since. However, the vector-valued condition is not convenient to work with computationally. To overcome such difficulty, in case of integral weights, Bruinier and Bundschuh [4] constructed an isomorphism between prime-level complex-valued modular forms and full-level vector-valued modular forms. Such an isomorphism proves to be useful, with which Choi [8] proved the Zagier duality for level 5, 13, 17, and Kim and Lee [10] provided automorphic corrections to some rank two hyperbolic Kac-Moody algebras. Later, following the work of

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Scheithauer ([15, 16]), the author generalized Bruinier and Bundschuh's isomorphism, proved the Zagier duality for general level, and with Kim and Lee, provided more automorphic corrections (see [24, 25, 11]).

On the half-integral weight side, it has been well-known that half-integral weight modular forms for $\Gamma_0(4)$ can be mapped to full-level vector-valued modular forms with two components, which can be made into an isomorphism if we choose the Kohnen's plus space. With such an isomorphism, Borchers' theorem in [1] is a special case of his automorphic product theorem in [2]. For the treatment via Jacobi modular forms, see Eichler and Zagier [9]. Our main purpose of this paper is to construct such an isomorphism on weakly holomorphic modular forms. Actually, in order to fit in Borchers' theory, we construct an explicit isomorphism between certain spaces of scalar-valued modular forms and modular forms for Weil representations:

Main Theorem: Assume that D is an anisotropic discriminant form that corresponds to the triple (N, χ', ϵ) where $N = 4M$ with M odd and square-free, $\chi' = 1$ and $\epsilon = (\epsilon_p)_{p|N}$ with $\epsilon_p = \pm 1$. Let $f = \sum_n a_n(y)e(nx)$ be a real analytic modular form of level N , weight k and trivial character that satisfies the ϵ -condition. Then $f \mapsto F = \sum_\gamma F_\gamma \mathbf{e}_\gamma$ with

$$F_\gamma(\tau) = \sum_{n \equiv NQ(\gamma) \pmod{NZ}} s(n) a_n(y/N) e(nx/N)$$

defines an isomorphism to the space of $\text{Aut}(D)$ -invariant modular forms of weight k and type ρ_D . (This is Corollary 5.6 and see (5) and (6) for the meaning of the ϵ -condition and the definition of $s(n)$ respectively.)

We actually work with slightly more general anisotropic discriminant forms (see Theorem 5.3). For example, the p -component D_p can be $p^{\pm 2}$ with $\pm = -\left(\frac{-1}{p}\right)$, which is not cyclic. We emphasize that the actual character χ coming from D differs from χ' for f by the Kronecker symbol $\left(\frac{N}{\cdot}\right)$ (see Remark 4.4 (ii)). The Atkin-Lehner operators $Y(p)$ and $Y(4)$ decompose the space of modular forms into common eigenspaces. More precisely, Kohnen's plus space condition corresponds to an eigenvalue of $Y(4)$ as expected and for a component $p^{\pm 1}$ or equivalently $\chi_p \neq 1$, the two eigenvalues are $\pm \epsilon_p^{-1} \sqrt{p}$, so the corresponding eigenspaces can be labelled by the sign. For a component $p^{\pm 2}$, it turns out that the eigenvalue -1 is the right choice for the isomorphism and the eigenvalue p does not play a role. Therefore, our total space of modular forms will be the common eigenspace for $Y(4)$ and for all of $Y(p)$ with $D_p = p^{\pm 2}$ with prescribed eigenvalues, and to include all other operators $Y(p)$ with $D_p = p^{\pm 1}$ we specify a sign vector that contains a sign for each such p . This treatment shows some similarity between $Y(4)$ and $Y(p)$ with $\chi_p = 1$, and for components $p^{\pm 2}$, the eigenvalue -1 gives the *new* space in the sense that it is mapped to full-level (vector-valued) modular forms. This provides a characterization of the eigenspaces of the $Y(p)$ operator when $\chi_p = 1$, which is missing from the literature (see [20]). The proof is similar to the integral-weight case and we need the concrete formulas for the matrix coefficients of the Weil representations obtained by Strömberg [19].

After establishing the isomorphism, we move on to prove the corresponding Zagier duality (Theorem 6.2) and write down the Borchers theorem in the case of $O(2, 1)$ explicitly (Theorem 6.3). For simplicity, we shall assume that the components $p^{\pm 2}$ do not appear. By considering

reduced modular forms and employing the obstruction theorem of Borchers [3] as we did in [25], we can prove the Zagier duality with little effort. For Borchers' theorem, most of the parts are straightforward except the computation of the Weyl vector. It follows immediately from the proofs that the isomorphism above can be applied to meromorphic or real analytic modular forms, hence in particular to Zagier's non-holomorphic modular form \mathbf{G} of weight $3/2$. The corresponding vector-valued modular forms \mathbf{G}_N for \mathbf{G} (and the invariant theta series) satisfies Lemma 9.5 and Corollary 9.6 of [2], from which the explicit formula for the Weyl vector follows. We caution the reader that Lemma 9.5 therein does not determine \mathbf{G}_N but its proof does. This is also observed by Bruinier and Schwagenscheidt in [6] (see their Remark 5.5), where the authors provide formulas for the Weyl vectors at various cusps using the theory of harmonic Maass forms.

Here is the layout of this paper: after providing the basics in Section 1, we classify anisotropic discriminant forms in Section 2. In Section 3, we briefly cover the Atkin-Lehner operators and the corresponding eigenspaces. In Section 4, under the assumption $D_2 = 2_{\pm 1}^{+1}$, we describe the ϵ -condition that is needed for the isomorphism and establish the isomorphism in Section 5. In Section 6, assuming that $D_p = p^{\pm 1}$ for all $p \mid M$, we prove the Zagier duality and translate Borchers' theorem. Finally we construct some examples in the last section.

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1. PRELIMINARIES

We recall some basic materials on discriminant forms and modular forms. For more details on discriminant forms, one may consult [14, 15].

1.1. Discriminant forms. A discriminant form is a finite abelian group D with a quadratic form $Q : D \rightarrow \mathbb{Q}/\mathbb{Z}$, such that the symmetric bilinear form defined by $(\beta, \gamma) = Q(\beta + \gamma) - Q(\beta) - Q(\gamma)$ is nondegenerate, namely, the map $D \rightarrow \text{Hom}(D, \mathbb{Q}/\mathbb{Z})$ defined by $\gamma \mapsto (\gamma, \cdot)$ is an isomorphism. We shall also write $Q(\gamma) = \frac{\gamma^2}{2}$. For $n \in \mathbb{Q}/\mathbb{Z}$, we say D represents n if its quadratic form represents n , namely if n is the norm of an element. The level of D is defined to be the smallest positive integer N such that $NQ(\gamma) = 0$ for each $\gamma \in D$. It is well-known that if L is an even lattice then L'/L is a discriminant form, where L' is the dual lattice of L . Conversely, any discriminant form can be obtained this way. With this, the signature $\text{sign}(D) \in \mathbb{Z}/8\mathbb{Z}$ is defined to be the signature of L modulo 8 for any even lattice L such that $L'/L = D$.

Every discriminant form can be decomposed into a direct sum of Jordan p -components for primes p and each Jordan p -component can be written as a direct sum of indecomposable Jordan q -components with q powers of p . Such decompositions are not unique in general. To fix our notations, we recall the possible indecomposable Jordan q -components as follows.

Let p be an odd prime and $q > 1$ be a power of p . The indecomposable Jordan components with exponent q are denoted by q^{δ_q} with $\delta_q = \pm 1$, and they are cyclic groups of order q with a generator γ , such that $Q(\gamma) = \frac{a}{q}$ and $\delta_q = \left(\frac{2a}{p}\right)$. These discriminant forms both have level q .

If $q > 1$ is a power of 2, there are also precisely two indecomposable *even* Jordan components of exponent q , denoted $q^{\delta_q 2} = q_{II}^{\delta_q 2}$ with $\delta_q = \pm 1$, and they are direct sums of two cyclic groups of order q , generated by two generators γ, γ' , such that if $\delta_q = 1$, we have

$$Q(\gamma) = Q(\gamma') = 0, \quad (\gamma, \gamma') = \frac{1}{q},$$

and if $\delta_q = -1$, we have

$$Q(\gamma) = Q(\gamma') = \frac{1}{q}, \quad (\gamma, \gamma') = \frac{1}{q}.$$

These two components both have level q . There are also *odd* indecomposable Jordan components in this case, denoted by $q_t^{\pm 1}$ with $\pm 1 = \left(\frac{2}{t}\right)$ for each $t \in (\mathbb{Z}/8\mathbb{Z})^\times$. Explicitly, $q_t^{\pm 1}$ is a cyclic group of order q with a generator γ such that $Q(\gamma) = \frac{t}{2q}$. Clearly, these discriminant forms have level $2q$.

Note that the symbols $q_t^{\delta_q n}$ represent discriminant forms, consistent with the notations in [15], not p -adic lattices as employed in [19, 23]. To give a finite direct sum of indecomposable Jordan components of the same exponent q , we multiply the signs, add the ranks, and add all subscripts t ($t = 0$ if there is no subscript). So in general, the q -component of a discriminant form is given by $q_t^{\delta_q n}$ ($t = 0$ if q is odd or the form is even). Set $k = k(q_t^{\delta_q n}) = 1$ if q is not a square and $\delta_q = -1$, and 0 otherwise. If q is odd, then define p -excess($q_t^{\pm n}$) = $n(q - 1) + 4k \pmod{8}$, and if q is even, then define oddity($q_t^{\pm n}$) = 2-excess($q_t^{\pm n}$) = $t + 4k \pmod{8}$.

Let D be a discriminant form and assume that D has a Jordan decomposition $D = \bigoplus_q q_t^{\delta_q n_q}$ where the sum is over distinct prime powers q . Then

$$p\text{-excess}(D) = \sum_{q:p|q} p\text{-excess}(q_t^{\delta_q n_q}).$$

We recall the well-known *oddity formula*:

$$\text{sign}(D) + \sum_{p>2} p\text{-excess}(D) = \text{oddity}(D) \pmod{8}.$$

1.2. Metaplectic covers. Throughout this note, $k \in \frac{1}{2} + \mathbb{Z}$ and \mathbb{H} denotes the upper half plane. For a non-zero complex number z , the square root \sqrt{z} or $z^{\frac{1}{2}}$ will be taken in the principal branch, that is $\arg(\sqrt{z}) \in (-\pi/2, \pi/2]$. Moreover, for an integer m , $z^{\frac{m}{2}}$ will mean $(\sqrt{z})^m$.

Let $\widetilde{\text{GL}}_2^+(\mathbb{R})$ be the metaplectic cover of $\text{GL}_2^+(\mathbb{R})$, so a typical element in $\widetilde{\text{GL}}_2^+(\mathbb{R})$ is of the form (A, ϕ) where ϕ is a holomorphic function on \mathbb{H} and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{R}), \quad \phi(\tau) = tj(A, \tau), \text{ for some } t \in \mathbb{C}, |t| = 1.$$

Here we follow the notation in [18] and denote $j(A, \tau) = \det(A)^{-\frac{1}{4}}(c\tau + d)^{\frac{1}{2}}$, so it is the square root of the usual automorphy factor in the case of integral weights. The group multiplication is defined by

$$(A, \phi)(B, \psi) := (AB, \phi(B\tau)\psi(\tau)), \quad (A, \phi), (B, \psi) \in \widetilde{\text{GL}}_2^+(\mathbb{R}).$$

The multiplier system ν on $\Gamma_0(4)$ is given by

$$\nu(A) = \left(\frac{c}{d}\right) \varepsilon_d^{-1}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4),$$

where $\left(\frac{c}{d}\right)$ is the Kronecker symbol and $\varepsilon_d = 1$ if $d \equiv 1 \pmod{4}$ and i otherwise. We recall that if d is odd and positive, this is the usual Jacobi symbol; $\left(\frac{c}{d}\right) = 0$ if $\gcd(c, d) > 1$; $\left(\frac{c}{d}\right) = \text{sign}(c) \left(\frac{-c}{d}\right)$ if $cd \neq 0$; $\left(\frac{2}{d}\right) = \left(\frac{d}{2}\right)$ and $\left(\frac{0}{\pm 1}\right) = \left(\frac{\pm 1}{0}\right) = 1$. These conditions determine the symbol by complete multiplicativity in d . Note that for $A \in \Gamma_0(4)$,

$$\bar{\nu}(A) = \nu^3(A) = \left(\frac{-1}{d}\right) \nu(A), \quad \nu(A)\nu(A^{-1}) = 1.$$

For any $A \in \text{GL}_2^+(\mathbb{R})$, we set

$$\tilde{A} = (A, j(A, \tau)) \in \widetilde{\text{GL}}_2^+(\mathbb{R}),$$

a specific lift of A to $\widetilde{\text{GL}}_2^+(\mathbb{R})$. Moreover, if $A \in \Gamma_0(4)$, we denote

$$A^* = (A, \nu(A)j(A, \tau)).$$

It is well-known that $A \mapsto A^*$ gives an injective homomorphism $\Gamma_0(4) \rightarrow \widetilde{\text{GL}}_2^+(\mathbb{R})$ and we denote its image by $\Delta_0(4)$. The image of $\Gamma_0(N)$ for $4 \mid N$ will be denoted by $\Delta_0(N)$ and that of $\Gamma_1(N)$ by $\Delta_1(N)$.

Let $\text{Mp}_2(\mathbb{Z})$ be the metaplectic double cover of $\text{SL}_2(\mathbb{Z})$ inside $\widetilde{\text{GL}}_2^+(\mathbb{R})$, consisting of pairs (A, ϕ) with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ and $\phi^2 = c\tau + d$. Let S and T denote the standard generators of $\text{SL}_2(\mathbb{Z})$, so

$$\tilde{S} = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right), \quad \tilde{T} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right)$$

generate $\text{Mp}_2(\mathbb{Z})$. We shall also need

$$Z := \tilde{I} = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right),$$

and we have $Z^4 = \tilde{I}$ and $\tilde{S}^2 = Z$.

1.3. Modular forms. Let $(A, \phi) \in \widetilde{\text{GL}}_2^+(\mathbb{R})$ and f be a function on \mathbb{H} . The weight- k slash operator is defined by

$$(f|_k(A, \phi))(\tau) = \phi^{-2k}(\tau) f(A\tau), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We shall usually drop the weight k from this notation.

Given any discriminant form D , let r denote the signature of D and let $\{\mathbf{e}_\gamma : \gamma \in D\}$ be the standard basis of the group algebra $\mathbb{C}[D]$. The Weil representation ρ_D attached to D is a unitary representation of $\text{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[D]$ such that

$$\begin{aligned} \rho_D(\tilde{T})\mathbf{e}_\gamma &= e(Q(\gamma))\mathbf{e}_\gamma, \\ \rho_D(\tilde{S})\mathbf{e}_\gamma &= \frac{i^{-\frac{r}{2}}}{\sqrt{|D|}} \sum_{\beta \in D} e(-\langle \beta, \gamma \rangle)\mathbf{e}_\beta, \end{aligned}$$

where $e(x) = e^{2\pi ix}$ and $|D|$ is the order of D . In particular, we have $\rho_D(Z)\mathbf{e}_\gamma = i^{-r}\mathbf{e}_{-\gamma}$. For convenience, we shall also denote $e_m(x) = e^{\frac{2\pi ix}{m}}$.

Denote by $\text{Aut}(D)$ the automorphism group of D , that is, the group of group automorphisms of D that preserve the quadratic form. The action of $\text{Aut}(D)$ commutes with ρ_D on $\mathbb{C}[D]$. We caution here that our ρ_D is the same as that in [2, 4], but conjugate to that in [15, 16].

Let $\mathcal{A}(k, \rho_D)$ be the space of functions $F = \sum_{\gamma \in D} F_\gamma \mathbf{e}_\gamma$ on \mathbb{H} , valued in $\mathbb{C}[D]$, such that

- $F|A := \sum_{\gamma} F_\gamma |A \mathbf{e}_\gamma = \rho_D(A)F$ for all $A \in \text{Mp}_2(\mathbb{Z})$,
- F is holomorphic on \mathbb{H} and meromorphic at ∞ ; namely, for each $\gamma \in D$, F_γ is holomorphic on \mathbb{H} and has Fourier expansion at ∞ with at most finitely many negative power terms.

More explicitly, if $F = \sum_{\gamma} F_\gamma \mathbf{e}_\gamma \in \mathcal{A}(k, \rho_D)$, then

$$F_\gamma(\tau) = \sum_{n \in Q(\gamma) + \mathbb{Z}, n \gg -\infty} a(\gamma, n) q^n.$$

Denote by $\mathcal{M}(k, \rho_D)$ and $\mathcal{S}(k, \rho_D)$ the subspace of holomorphic modular forms and the subspace of cusp forms, respectively. Because the action of $\text{Aut}(D)$ and that of ρ_D commute on $\mathbb{C}[D]$, the spaces of vector-valued modular forms will be stable under $\text{Aut}(D)$. We define $\mathcal{A}^{\text{inv}}(k, \rho_D)$ to be the subspace of functions that are invariant under $\text{Aut}(D)$. Since $\gamma \mapsto -\gamma$ gives an element in $\text{Aut}(D)$, it is clear that $F_\gamma = F_{-\gamma}$ for $F \in \mathcal{A}^{\text{inv}}(k, \rho_D)$. It thus follows from the action of Z that the space $\mathcal{A}^{\text{inv}}(k, \rho_D)$ is trivial unless $2k \equiv r \pmod{4}$. Similarly, we define $\mathcal{M}^{\text{inv}}(k, \rho_D)$ and $\mathcal{S}^{\text{inv}}(k, \rho_D)$.

For each Dirichlet character χ modulo N , denote by $\mathcal{A}(N, k, \chi)$ the space of holomorphic functions f on \mathbb{H} such that $f|A^* = \chi(A)f$ for each $A \in \Gamma_0(N)$ and f is meromorphic at cusps. By considering $A = -I$, we see that for the space to be non-zero, we necessarily have $\chi(-1) = 1$. The subspace of holomorphic forms and that of cuspforms are denoted by $\mathcal{M}(N, k, \chi)$ and $\mathcal{S}(N, k, \chi)$ respectively.

Each discriminant form D can be decomposed uniquely into p -components $D = \bigoplus_p D_p$ and each Dirichlet character χ can also be decomposed uniquely into p -components $\chi = \prod_p \chi_p$. For each positive integer m , we shall denote $D_m = \bigoplus_{p|m} D_p$ and $\chi_m = \prod_{p|m} \chi_p$ for convenience.

2. ANISOTROPIC DISCRIMINANT FORMS

Let D be an arbitrary discriminant form. It is trivial that if $\text{Aut}(D)$ acts transitively on each subset of elements of fixed norm, then D is anisotropic, namely the only element in D with norm 0 is 0. In this section, we prove the converse and classify anisotropic discriminant forms in general, and then prove some of their properties. Similar treatment for square-free levels was done in [7].

Proposition 2.1. A discriminant form D is anisotropic if and only if $\text{Aut}(D)$ acts transitively on each subset of elements of fixed norm, and if and only if $D = \bigoplus_p D_p$ such that

- for an odd prime p , D_p is either trivial or equal to $p^{\pm 1}$, or $D_p = p^{+2}$ when $p \equiv 3 \pmod{4}$, or $D_p = p^{-2}$ when $p \equiv 1 \pmod{4}$;

- D_2 is either trivial or equal to one of the following:

$$2_{\pm 3}^{+3}, \quad 2_{\pm 2}^{+2}, \quad 2_{\pm 1}^{+1}, \quad 2^{-2}, \quad 4_t^{\pm 1}, \quad 4_t^{\pm 1} \oplus 2_{\pm 1}^{+1}.$$

Proof. Assume that $D = \bigoplus_p D_p$ is anisotropic, so D_p and all of its indecomposable components are anisotropic. We first claim that any indecomposable component $q_t^{\delta_p n}$ is equal to either $p^{\pm 1}$ for some odd prime p or one of the following: $2_{\pm 1}^{+1}$, 2^{-2} , $4_t^{\pm 1}$. Indeed, when p is odd and $q = p^f$ with $f \geq 2$, then $p^{f-1} + q\mathbb{Z}$ is a non-zero element with zero norm, which is not possible. The claim on the 2-components follows similarly, for which we should note that $2_{\pm 1}^{+1} \cong 2_{\pm 3}^{-1}$. Now we prove that D_p is of the form in the statement. If p is odd, we have just seen that $D_p = p^{\delta_p n_p}$ for some $\delta_p \in \{\pm 1\}$ and $n_p \geq 0$. If $n_p \geq 3$, by Corollary 2 on page 6 of [17], D_p is isotropic. If $n_p = 2$ and $p \equiv 1 \pmod{4}$, then $-1 \pmod{p}$ is a square. By Corollary 2 on page 33 of [17], since D is anisotropic, the two indecomposable components represent different elements. So the two components have different signs and $\delta_p = -1$. The case when $p \equiv -1 \pmod{4}$ follows in the same way. When $p = 2$, we note that

$$(1) \quad 2^{-2} \oplus 2_{\pm 1}^{+1} \cong 2_{\mp 3}^{+3}, \quad 4_t^{\pm 1} \oplus 2_{\pm 1}^{+1} \cong 4_t^{\pm 1} \oplus 2_{\mp 1}^{+1}.$$

Moreover, 2^{-2} and $4_t^{\pm 1}$ together appear at most once, so this case follows from the isomorphisms in (1). Conversely, we see easily that each D_p in the list is anisotropic.

We then prove that being anisotropic implies the transitivity of the action of $\text{Aut}(D)$. When $D_p = p^{\delta_p n_p}$ for an odd p , since D_p is anisotropic, any two nonzero elements γ, γ' with the same norm generate two non-degenerate one-dimensional \mathbb{F}_p -subspaces, so $\gamma \mapsto \gamma'$ extends to an element in $\text{Aut}(D_p)$ by Witt's theorem. This shows that the action of $\text{Aut}(D_p)$ is transitive. That those D_2 in question are also transitive follows from explicit computation. We treat the case $D_2 = 2_{+3}^{+3}$ and leave other cases to the reader. We see that D is generated by three elements $\gamma_1, \gamma_2, \gamma_3$ of norm $\frac{1}{4}$. There are three elements $\gamma_1 + \gamma_2, \gamma_1 + \gamma_3, \gamma_2 + \gamma_3$ with norm $\frac{1}{2}$ and the other two elements have norm 0 and $\frac{3}{4}$ respectively. We see that as a permutation group on the three generators, $\text{Aut}(D)$ is isomorphic to the symmetric group S_3 . In particular, $\text{Aut}(D)$ acts transitively on the set of elements of norm $\frac{1}{4}$ and that of elements of norm $\frac{1}{2}$. Since $\text{Aut}(D) = \bigoplus_p \text{Aut}(D_p)$, the action of $\text{Aut}(D)$ on each subset of fixed norm is also transitive. This completes the proof. \square

We prove a lemma on the representing behavior.

Lemma 2.2. Let p be an odd prime and $D = p^{\pm n}$ be anisotropic.

- (i) If $n = 1$, D represents each non-zero norm exactly 2 times.
- (ii) If $n = 2$, D represents each non-zero element of $\frac{1}{p}\mathbb{Z}/\mathbb{Z}$ exactly $p + 1$ times.

Proof. By identifying $\frac{1}{p}\mathbb{Z}/\mathbb{Z}$ with $\mathbb{Z}/p\mathbb{Z}$, D becomes a quadratic form over \mathbb{F}_p . If $n = 1$, then it is clear that D represents either quadratic residues or quadratic non-residues modulo p but not both, and (i) follows easily.

If $n = 2$ and a is a norm, let D^a be the subset of elements with norm a . So $D^a \neq \emptyset$ and we need to prove that $|D^a| = p + 1$. Fix any $\gamma_0 \in D^a$ and consider the map $h: \text{Aut}(D) \rightarrow D^a$ given by $\sigma \mapsto \sigma(\gamma_0)$. For any $\gamma \in D^a$, $\gamma_0 \mapsto \gamma$ extends to an element $\sigma \in \text{Aut}(D)$ by Witt's theorem

(see page 31 of [17]), showing that h is surjective. Moreover, upon choosing the orthogonal complements

$$D = \mathbb{F}_p \gamma_0 \perp \mathbb{F}_p \gamma'_0 = \mathbb{F}_p \gamma \perp \mathbb{F}_p \gamma',$$

there are precisely two possibilities for $\sigma(\gamma'_0)$ as seen in (i). It follows that h is a 2-to-1 map, hence $2|D^a| = |\text{Aut}(D)|$, which is therefore independent of a . By Corollary 2 on page 6 and Corollary 1 on page 33 of [17], D represents everything, so the $p - 1$ non-zero norms are equally distributed over the $p^2 - 1$ non-zero elements of D . It follows that there are exactly $p + 1$ elements with norm a , for each $a \in \mathbb{F}_p^\times$, finishing the proof. \square

The following lemma is crucial in proving our isomorphism later and we prove it in a similar way as in [24] with minor modifications.

Lemma 2.3. Let D be anisotropic and for a fixed modular form $F = \sum_{\gamma} F_{\gamma} \mathbf{e}_{\gamma} \in \mathcal{A}^{\text{inv}}(k, \rho_D)$, let $W = \text{span}_{\mathbb{C}} \{F_{\gamma} : \gamma \in D\}$. Then

- (i) $W = \text{span}_{\mathbb{C}} \{F_0 | A : A \in \text{Mp}_2(\mathbb{Z})\}$. In particular, if $F_0 = 0$, then $F = 0$.
- (ii) If $f = \sum_{\gamma \in D} a_{\gamma} F_{\gamma}$ is T -invariant, then $f = a_0 F_0$.

Proof. Denote by W' the space spanned by $F_0 | A$, $A \in \text{Mp}_2(\mathbb{Z})$ and we need to prove $W = W'$. Note that $F | A = \rho_D(A) F$, so

$$(2) \quad F_0 | A = (\rho_D(A) F, \mathbf{e}_0) = \sum_{\gamma} F_{\gamma} (\rho_D(A) \mathbf{e}_{\gamma}, \mathbf{e}_0).$$

This implies that $W' \subset W$. To prove the other inclusion, we only have to show that $F_{\gamma} \in W'$ for each $\gamma \in D$. Note that for each $\gamma \in D$, $(\rho_D(\tilde{S}) \mathbf{e}_{\gamma}, \mathbf{e}_0) = i^{-\frac{r}{2}} |D|^{-\frac{1}{2}} \neq 0$, which is independent of γ . It follows that $\sum_{\beta \in D} F_{\beta} \in W'$ by (2) with $A = \tilde{S}$. Since F is $\text{Aut}(D)$ -invariant and D is anisotropic, by Proposition 2.1, all γ with fixed norm n have equal function F_{γ} , which we denote by F_n . It follows that $\sum_n a_n F_n \in W'$, where $a(n)$ is equal to the number of $\gamma \in D$ with norm n . Clearly for each positive integer j ,

$$\sum_n a(n) F_n | \tilde{T} = \sum_n a(n) e(nj) F_n \in W'.$$

Since $e(n)$'s are mutually distinct, this implies that $F_n \in W'$ by the theory of Vandermonde matrices. This completes the proof of part (i).

Part (ii) follows directly from the fact that each F_{γ} is an eigenfunction of T and only F_0 has eigenvalue 1. \square

From now on, we shall always assume that D is anisotropic. By Proposition 2.1, the level N of D is the conductor of a quadratic Dirichlet character; namely, the 2-part of N is 1, 4 or 8 and the odd part is square-free. We will see in Lemma 4.1 that a primitive quadratic Dirichlet character is uniquely determined by D .

Remark 2.4. We need Lemma 2.3 (i) to show that F_0 determines F , but the assumption that D is anisotropic may not be necessary. For example, if $D = 2^{+2}$, then the condition that F_0 determines F still holds, but Lemma 2.3 (ii) is no longer true (see [2, Example 13.7]).

3. ATKIN-LEHNER OPERATORS

In this section, we consider the Atkin-Lehner operators on the space $\mathcal{A}(N, k, \chi)$, where $N = 4M$, M is odd square-free, $k \in \frac{1}{2} + \mathbb{Z}$, and χ is a quadratic Dirichlet character modulo N . Even though the half-integral weight case is similar to the integral weight case, we treat it in details for later computation. The main references are [12] and [20].

For any odd divisor m of N , we choose γ_m and γ_{4m} in $\mathrm{SL}_2(\mathbb{Z})$ such that

$$\gamma_m \equiv \begin{cases} S & \text{mod } m^2, \\ I & \text{mod } (\frac{N}{m})^2, \end{cases} \quad \gamma_{4m} := S\gamma_{M/m}^{-1} \equiv \begin{cases} S & \text{mod } (4m)^2, \\ I & \text{mod } (\frac{M}{m})^2. \end{cases}$$

The existence of γ_m , hence that of γ_{4m} , follows from the existence of such matrices in $\mathrm{SL}_2(\mathbb{Z}/N^2\mathbb{Z})$ by the Chinese Remainder Theorem and then from the surjectivity of $\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/N^2\mathbb{Z})$. When $m = 1$, we simply take $\gamma_m = I$. If $m > 1$, clearly all of entries of γ_m are non-zero and we shall assume for simplicity that they are positive; this can be achieved by left and/or right multiplication by matrices in $\Gamma(N^2)$ as follows: given $\gamma_m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, right multiplication by $\begin{pmatrix} 1-2cdN^2 & -dN^2 \\ 4c^2dN^2 & 1+2cdN^2 \end{pmatrix}$ makes the left-lower entry positive, right multiplication by powers of $\begin{pmatrix} 1 & N^2 \\ 0 & 1 \end{pmatrix}$ makes the lower entries positive, and finally left multiplication by powers of $\begin{pmatrix} 1 & 0 \\ N^2 & 1 \end{pmatrix}$ makes all entries positive. Actually, for later computations it is enough to assume that the entries a, c, d are positive.

For any nonzero integer m , let

$$\delta_m = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{\delta}_m = \left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}, m^{-\frac{1}{4}} \right).$$

For any odd positive divisor m of N , let $W(m) = \gamma_m^* \widetilde{\delta}_m$. which makes sense since m is odd and $\gamma_m \in \Gamma_0(4)$. For the Fricke involution, we follow Shimura's notation and define

$$(3) \quad \beta_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}, \quad \tau_N = (1, \sqrt{-i}) \widetilde{\beta}_N = \left(\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}, N^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}} \right).$$

To make it clear that $W(m)$ is defined only for odd m , the modern notation W_N for Fricke involution is not adopted. Direct computation shows that $W(m)$ and τ_N normalize $\Delta_1(N)$, the image of $\Gamma_1(N)$ under the isomorphism $\Gamma_0(4) \cong \Delta_0(4)$.

Let $\xi \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that $\Delta_1(N)$ and $\xi\Delta_1(N)\xi^{-1}$ are commensurable, namely $\Delta_1(N)\xi\Delta_1(N)$ can be expressed as a finite disjoint union $\Delta_1(N)\xi\Delta_1(N) = \bigcup_{\nu} \Delta_1(N)\xi_{\nu}$. Then for a function f on \mathbb{H} that is invariant under the weight- k slash action of $\Delta_1(N)$, define

$$f|[\Delta_1(N)\xi\Delta_1(N)] := \det(\xi)^{\frac{k}{2}-1} \sum_{\nu} f|\xi_{\nu}.$$

For each divisor m , even or odd, of N , define $U(m)$ as follows:

$$f|U(m) = m^{\frac{k}{2}-1} \sum_{j \bmod m} f|\widetilde{\delta}_m^{-1} \widetilde{T}^j = m^{\frac{k}{2}-1} \sum_{j \bmod m} f|\widetilde{\delta}_m^{-1} T^j.$$

Finally, we define $Y(p)$ for each odd prime $p \mid N$ by

$$f|Y(p) = p^{1-\frac{k}{2}} f|U(p)W(p),$$

and $Y(4)$ by

$$f|Y(4) = 4^{1-\frac{k}{2}} f|U(4)W(M)\tau_N.$$

We collect a few properties of these operators in the following proposition:

Proposition 3.1. Let $f \in \mathcal{A}(N, k, \chi)$.

(i) $f|\tau_N \in \mathcal{A}(N, k, \chi(\frac{N}{\cdot}))$ and $f|\tau_N^2 = f$.

(ii) Let $m \mid N$ be odd or divisible by 4. Then $f|U(m) \in \mathcal{A}(N, k, \chi(\frac{m}{\cdot}))$ and

$$f|U(m) = m^{\frac{k}{2}-1} f|[\Delta_1(N)\widetilde{\delta}_m^{-1}\Delta_1(N)].$$

(iii) For each $m \mid M$, $f|W(m) \in \mathcal{A}(N, k, \chi(\frac{m}{\cdot}))$ and

$$f|W(m)^2 = \varepsilon_m^{-2k} \chi_m(-1) \chi_{N/m}(m) f.$$

Moreover, if $m, m' \mid M$ and $(m, m') = 1$, then $f|W(m)W(m') = \chi_{m'}(m) f|W(mm')$.

(iv) For any $m, m' \mid M$ with $(m, m') = 1$, then

$$f|W(m)U(m') = \chi_m(m') f|U(m')W(m) \text{ and } f|U(4)W(m) = f|W(m)U(4).$$

(v) For any $m \mid M$, we have

$$f|U(m)W(m) = m^{\frac{k}{2}-1} \prod_{p|m} \chi_p(m/p) \left(\frac{m/p}{p} \right) \cdot f \left| \prod_{p|m} Y(p) \right.$$

and

$$f|\tau_N U(m)W(m) = \chi_m(M/m) f|W(m)U(m)\tau_N.$$

Proof. Denote $\Delta_1 = \Delta_1(N)$. We assume that $m > 1$ since the corresponding statements are trivial if $m = 1$. (i) and (ii) are contained in Proposition 1.4 and 1.5 of [18], (iii) is Proposition 1.18 of [20], and the first part of (iv) can be obtained easily from (iii) and Proposition 1.20 (1) of [20]. For the second identity in (iv), for each prime $p \mid M$, since $W(p)$ normalizes Δ_1 ,

$$\begin{aligned} f|U(4)W(p) &= 4^{\frac{k}{2}-1} f|[\Delta_1\widetilde{\delta}_4^{-1}\Delta_1]W(p) = 4^{\frac{k}{2}-1} f|[\Delta_1\widetilde{\delta}_4^{-1}W(p)\Delta_1] \\ &= 4^{\frac{k}{2}-1} f|[\Delta_1\widetilde{\delta}_4^{-1}\gamma_p^*\widetilde{\delta}_p\Delta_1] = 4^{\frac{k}{2}-1} f|[\Delta_1(\delta_4^{-1}\gamma_p\delta_4)^*\widetilde{\delta}_p\widetilde{\delta}_4^{-1}\Delta_1] \\ &= 4^{\frac{k}{2}-1} f|[\Delta_1\alpha^*W(p)\widetilde{\delta}_4^{-1}\Delta_1], \end{aligned}$$

where $\alpha = \delta_4^{-1}\gamma_p\delta_4\gamma_p^{-1} \in \Gamma_0(N)$. Since χ is quadratic and the right-lower entry d_α of α satisfies $d_\alpha \equiv 4 \pmod{p}$ and $d_\alpha \equiv 1 \pmod{N/p}$, d_α is a square modulo N and we have $f|\alpha^* = f$. It follows that

$$f|U(4)W(p) = 4^{\frac{k}{2}-1} f|W(p)[\Delta_1\widetilde{\delta}_4^{-1}\Delta_1] = f|W(p)U(4).$$

The general case follows from part (iii) by decomposing $W(m)$.

The proof of the second part of (v) is similar:

$$\begin{aligned}
 f|\tau_N U(m)W(m) &= m^{\frac{k}{2}-1} f|[\Delta_1 \tau_N \widetilde{\delta_m^{-1}} W(m) \Delta_1] \\
 &= m^{\frac{k}{2}-1} f|[\Delta_1 (\tau_N \widetilde{\delta_m^{-1}} W(m) \tau_N^{-1}) \tau_N \Delta_1] \\
 &= m^{\frac{k}{2}-1} f|[\Delta_1 \alpha^* \tau_N \Delta_1] \\
 &= m^{\frac{k}{2}-1} f|(\alpha \gamma_m^{-1})^* [\Delta_1 \gamma_m^* \tau_N \Delta_1] \\
 &= f|(\alpha \gamma_m^{-1})^* W(m) U(m) \tau_N,
 \end{aligned}$$

where

$$\alpha \gamma_m^{-1} \equiv \begin{cases} \begin{pmatrix} (N/m)^{-1} & 0 \\ 0 & N/m \end{pmatrix} & \text{mod } m \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{mod } N/m \end{cases}$$

(the inverse is taken in $\mathbb{Z}/m\mathbb{Z}$). Then we have

$$f|\tau_N U(m)W(m) = \chi_m(M/m) f|W(m)U(m)\tau_N,$$

as expected. The first part of (v) follows from (iii) and (iv) and elementary computations; we omit the details but emphasize that one should apply the correct characters in the steps. \square

Now we consider the operators $Y(p)$ and $Y(4)$.

Proposition 3.2. (i) The space $\mathcal{A}(N, k, \chi)$ decomposes under $Y(4)$ into eigenspaces

$$\mathcal{A}(N, k, \chi) = \mathcal{A}(N, k, \chi)_{\mu_2^+} \oplus \mathcal{A}(N, k, \chi)_{\mu_2^-}$$

where the eigenvalues are

$$\mu_2^+ = \chi_2(-1)^{k+1} (-1)^{\lfloor \frac{2k+1}{4} \rfloor} 2^{\frac{3}{2}}, \quad \mu_2^- = -2^{-1} \mu_2^+.$$

Moreover, $f = \sum_n a(n)q^n \in \mathcal{A}(N, k, \chi)_{\mu_2^+}$ if and only if

$$a(n) = 0 \quad \text{whenever } \chi_2(-1)(-1)^{k-\frac{1}{2}n} \equiv 2, 3 \pmod{4}.$$

(ii) Assume that $p \mid M$ with $\chi_p = 1$, the space $\mathcal{A}(N, k, \chi)$ decomposes under $Y(p)$ into eigenspaces

$$\mathcal{A}(N, k, \chi) = \mathcal{A}(N, k, \chi)_{\mu_p^+} \oplus \mathcal{A}(N, k, \chi)_{\mu_p^-}$$

where the eigenvalues are $\mu_p^+ = \varepsilon_p^{-1} p^{\frac{1}{2}}$, $\mu_p^- = -\mu_p^+$. Moreover, $f = \sum_n a(n)q^n \in \mathcal{A}(N, k, \chi)_{\mu_p^\pm}$ if and only if

$$a(n) = 0 \quad \text{whenever } \left(\frac{n}{p}\right) = \mp 1.$$

(iii) Assume that $p \mid M$ with $\chi_p = \left(\frac{\cdot}{p}\right)$, the space $\mathcal{A}(N, k, \chi)$ decomposes under $Y(p)$ into eigenspaces

$$\mathcal{A}(N, k, \chi) = \mathcal{A}(N, k, \chi)_{\mu_p^+} \oplus \mathcal{A}(N, k, \chi)_{\mu_p^-}$$

where the eigenvalues are $\mu_p^+ = -1$, $\mu_p^- = p$.

(iv) The operators $Y(p)$, $p \mid M$, and $Y(4)$ map $\mathcal{A}(N, k, \chi)$ to itself and they all commute with one another. In particular, $\mathcal{A}(N, k, \chi)$ decomposes into a direct sum of common eigenspaces for these operators.

Proof. That $Y(p_1)$ and $Y(p_2)$ commute is given in Proposition 1.24 of [20], (i) follows from Proposition 1 of [12], (ii) from Proposition 1.29 of [20], and (iii) from Proposition 1.27 of [20] since $Y(p)$ satisfies the polynomial $X^2 - (p-1)X - p$.

We only have to prove that $Y(4)$ and $Y(p)$ commute, that is,

$$f|U(4)W(M)\tau_N U(p)W(p) = f|U(p)W(p)U(4)W(M)\tau_N.$$

Since $f|U(4)W(M) \in \mathcal{A}(N, k, \chi(\frac{M}{p}))$ and by Proposition 3.1 (v),

$$f|U(4)W(M)\tau_N U(p)W(p) = \chi_p(M/p) \left(\frac{M/p}{p} \right) f|U(4)W(M)W(p)U(p)\tau_N,$$

we only have to prove that

$$\chi_p(M/p) \left(\frac{M/p}{p} \right) f|U(4)W(M)W(p)U(p) = f|U(p)W(p)U(4)W(M).$$

This is done by decomposing $M = p \cdot \frac{M}{p}$ and applying Proposition 3.1 (iii) and (iv), while paying attention to the change of characters. \square

Now assume $\chi_p = 1$ for each $p \mid M$. Set $Z(p) = \varepsilon_p p^{-\frac{1}{2}} Y(p)$ for $p \mid M$, then $Z(p)$ is an involution on $\mathcal{A}(N, k, \chi)$ for every p . For each sign vector $\epsilon = (\epsilon_p)_{p \mid N}$, we denote $\mathcal{A}^\epsilon(N, k, \chi)$ to be the subspace in $\mathcal{A}(N, k, \chi)_{\mu_2^+}$ of modular forms f with $f|Z(p) = \epsilon_p f$ for all $p \mid M$. The following corollary computes the Fourier coefficients of the ϵ -components of a modular form in this particular setting:

Corollary 3.3. Assume that $\chi_p = 1$ for each $p \mid M$ and let $Z(p)$ be as above. If $f = \sum_n a(n)q^n \in \mathcal{A}(N, k, \chi)_{\mu_2^+}$, then $f = \sum_\epsilon f^\epsilon$ with $f^\epsilon = \sum_n b_\epsilon(n)q^n \in \mathcal{A}^\epsilon(N, k, \chi)$, where for $(n, N) = 1$,

$$b^\epsilon(n) = 2^{-\omega(M)} a(n) \prod_{p \mid M} \left(1 + \epsilon_p \left(\frac{n}{p} \right) \right).$$

Proof. It is clear that

$$f^\epsilon = 2^{-\omega(M)} f \Big| \prod_{p \mid M} (1 + \epsilon_p Z(p)),$$

and the corollary follows from Proposition 3.2 (ii). \square

4. DISCRIMINANT FORMS AND THE ϵ -CONDITION

Let D be an anisotropic discriminant form of odd signature r and level N .

Lemma 4.1. An anisotropic discriminant form D of odd signature r and level N determines an even quadratic Dirichlet character $\chi \bmod N$, such that

$$\rho_D(A)\mathbf{e}_0 = \nu(A)^r \chi(d)\mathbf{e}_0, \quad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Explicitly, if we write $\chi = \prod_p \chi_p$ into p -components, then if p is odd and $p \nmid d$,

$$\chi_p(d) = \begin{cases} 1, & p \nmid |D| \text{ or } p^2 \mid |D|, \\ \left(\frac{d}{p}\right), & \text{otherwise.} \end{cases}$$

For χ_2 and odd d , assuming $2 \mid |D|$,

$$\chi_2(d) = \begin{cases} 1, & \left(\frac{-1}{|D|}\right) = +1, D_2 = 2_{\pm 3}^{+3}, 2_{\pm 1}^{+1}, \\ \left(\frac{-1}{d}\right), & \left(\frac{-1}{|D|}\right) = -1, D_2 = 2_{\pm 3}^{+3}, 2_{\pm 1}^{+1}, \\ \left(\frac{2}{d}\right), & \left(\frac{-1}{|D|}\right) = +1, D_2 = 4_{\pm 1}^{+1}, 4_{\pm 3}^{-1}, \\ \left(\frac{-2}{d}\right), & \left(\frac{-1}{|D|}\right) = -1, D_2 = 4_{\pm 1}^{+1}, 4_{\pm 3}^{-1}. \end{cases}$$

Proof. That D determines a quadratic Dirichlet character χ follows from [19, Lemma 5.6] and for d coprime to N we obtain the explicit formula

$$\chi(d) = \begin{cases} \left(\frac{d}{2|D|}\right), & \left(\frac{-1}{|D|}\right) = 1, \\ \left(\frac{d}{2|D|}\right) \left(\frac{-1}{d}\right), & \left(\frac{-1}{|D|}\right) = -1, \end{cases}$$

by elementary computation and by the relation

$$\text{oddity}(D) \equiv \text{sign}(D) + \left(\frac{-1}{|D|}\right) - 1 \pmod{4}.$$

The formulas for local components χ_p then follow easily. Note that because the signature of D is odd, the other possibilities of D_2 in Proposition 2.1 do not appear. (This lemma also follows from the formulas in [23] which treats p -adic lattices.) \square

In order to introduce the ϵ -condition, we investigate the representing behaviour of D .

Lemma 4.2. Let $D = \bigoplus_p D_p$ be of level $N = \prod_p N_p$ and denote $N'_p = N/N_p$. Then for any integer n , D represents $\frac{n}{N}$ if and only if D_p represents $\frac{N'_p n}{N_p}$ for each $p \mid N$.

Proof. For $\gamma = \sum_p \gamma_p$ with $\gamma_p \in D_p$, we have $Q(\gamma) = \sum_p Q(\gamma_p)$. Now if b_p is the inverse of N'_p mod N_p , then $\frac{1}{N} + \mathbb{Z}$ decomposes p -adically as $\sum_p \frac{b_p}{N_p} + \mathbb{Z}$. Therefore, D represents $\frac{n}{N}$ if and only if D_p represents $\frac{b_p n}{N_p}$. It is easy to see that $Q(\gamma) = \frac{b_p n}{N_p}$ if and only if $Q(N'_p \gamma) = \frac{N'_p n}{N_p}$, so the lemma follows. \square

To simplify the exposition, we shall assume that $D_2 = 2_t^{+1}$, $t = \pm 1$ for the rest of this paper. The other cases are similar but computationally more complicated; see for example the even signature situation in [25]. Now the oddity formula says

$$(4) \quad \chi_2(-1) = \chi_M(-1) = e_4(r - t) = \left(\frac{-1}{|D|}\right),$$

where $N = 4M$, M odd and square-free, r the signature, and $|D|$ and N share the same set of primes divisors but are different at 2 and at p where $D_p = p^{\pm 2}$.

We explain how the data for the two sides, vector-valued and scalar-valued, correspond as follows. We begin with D , and it determines $N = 4M$ with M odd square-free and an even χ .

We construct a sign vector $\epsilon = (\epsilon_p)_p$ over $p = 2$ or $p \mid M$ such that $\chi_p \neq 1$ as follows: if $D_p = p^{\delta_p}$ with $\delta_p = \pm 1$, then we define $\epsilon_p = \chi_p(2M/p)\delta_p$; let $\epsilon_2 = t\left(\frac{-1}{N}\right)$ if $D_2 = 2_t^{+1}$. Therefore, D determines $(N, \chi\left(\frac{N}{\cdot}\right), \epsilon)$ where $N = 4M$, χ is an even Dirichlet character modulo N and ϵ is a sign vector. Given such a triple $(N, \chi\left(\frac{N}{\cdot}\right), \epsilon)$, since D_p for p with $\chi_p = 1$ is fixed by Proposition 2.1, the reconstruction of D is straightforward and based on D_2 being $2_{\pm 1}^{+1}$.

Following Shimura [18], we shall denote $\chi' = \chi\left(\frac{N}{\cdot}\right)$ from now on. The following lemma describes the set of norms $q(D)$ from the data (N, χ', ϵ) .

Lemma 4.3. Let D be as above and corresponding to (N, χ', ϵ) . Then D represents $\frac{n}{N}$ if and only if $n \equiv 0$ or $\epsilon_2 \pmod{4}$ and for each odd prime divisor p of the conductor χ , $\chi_p(n) \in \{0, \epsilon_p\}$.

Proof. Let c denote the odd part of the conductor of χ . By Lemma 4.2, D represents n/N if and only if D_p represents $\frac{nN/p}{p}$ for each odd $p \mid N$ and D_2 represents $\frac{nN/4}{4}$. Since if $p \nmid 2c$ and $p \mid N$, $D_p = p^{\pm 2}$ represents everything by Lemma 2.2, we only have to consider primes $p \mid 2c$. If $p \mid c$, then $D_p = p^{\delta_p}$, and from the definition of δ_p in Section 1, we see that D_p represents $\frac{nN/p}{p}$ if and only if $p \mid n$ or $\chi_p(2nN/p) = \delta_p$, hence if and only if $\chi_p(n) = 0$ or ϵ_p . Finally, $D_2 = 2_t^{+1}$ represents $\frac{nN/4}{4}$ if and only if $\frac{nN}{4} \equiv 0$ or $t \pmod{4}$, hence if and only if $n \equiv 0$ or $\epsilon_2 \pmod{4}$. This finishes the proof. \square

Given any data (N, χ', ϵ) as above, we define the associated modular form space $\mathcal{A}^\epsilon(N, k, \chi')$ to be the common eigenspace with eigenvalues μ_2^+ for $Y(4)$, $\mu_p^{\epsilon_p}$ for $Y(p)$ if $\chi_p \neq 1$ and $\mu_p^+ = -1$ for $Y(p)$ if $\chi_p = 1$. Explicitly, $f = \sum_n a(n)q^n \in \mathcal{A}^\epsilon(N, k, \chi')$ if and only if

$$(5) \quad \begin{aligned} a(n) &= 0, & \text{if } n \equiv 2, -\epsilon_2 \pmod{4}, \\ f|_k Y(p) &= -f, & \text{if } \chi_p = 1, \text{ and} \\ a(n) &= 0, & \text{if } \left(\frac{n}{p}\right) = -\epsilon_p \text{ for some } p \mid M, \chi_p \neq 1. \end{aligned}$$

We call (5) the ϵ -condition.

Remark 4.4. (i) *Kohnen's plus condition* for the space $\mathcal{A}(N, k, \chi')$ with $\chi' = \chi\left(\frac{N}{\cdot}\right)$ is given by $a(n) = 0$ if $\chi'_2(-1)(-1)^{k-\frac{1}{2}}n \equiv 2, 3 \pmod{4}$. Since $2k \equiv r \pmod{4}$, $\chi'_2(-1) = \chi_2(-1)\left(\frac{-1}{N}\right)$ and $\epsilon_2 = t\left(\frac{-1}{N}\right)$,

$$\epsilon_2 \chi'_2(-1)(-1)^{k-\frac{1}{2}} = t \chi_2(-1) e_4(2k-1) = \chi_2(-1) e_4(2k-1) e_4(1-t) = \chi_2(-1) e_4(r-t) = 1,$$

by (4). Therefore, Kohnen's plus condition on $\mathcal{A}(N, k, \chi')$ is the same as our ϵ_2 -condition for the data (N, k, χ') that corresponds to D .

(ii) On $\mathcal{A}^\epsilon(N, k, \chi')$, the appearance of χ' rather than χ may seem confusing, but this is the right choice to transfer the representing behaviour of D to the non-vanishing properties of Fourier coefficients. For the same reason, τ_N will be introduced in the constructions of the linear maps between modular form spaces in Section 5.

5. THE ISOMORPHISM: CONSTRUCTION AND THE PROOF

Let $N = 4M$ with M odd and squarefree and D be anisotropic with $D_2 = 2_{\pm 1}^{\pm 1}$. Let χ be an even quadratic Dirichlet character modulo N , we set $\chi' = \left(\frac{N}{\cdot}\right)\chi$. As always, we assume $2k \equiv r \pmod{4}$.

Lemma 5.1. Assume that D has level N and character χ . Then

(i) If $F \in \mathcal{A}(k, \rho_D)$, then $F_0|_{\tau_N} \in \mathcal{A}(N, k, \chi')$.

(ii) If $f \in \mathcal{A}(N, k, \chi')$, then

$$F = \sum_{A \in \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z})} \left(f|_{\tau_N \tilde{A}} \right) \rho_D(\tilde{A})^{-1} \mathbf{e}_0 \in \mathcal{A}^{\mathrm{inv}}(k, \rho_D).$$

Proof. For (i): By Proposition 1.4 of [18], we only have to prove that $F_0 \in \mathcal{A}(N, k, \chi)$, which in turn follows directly from Lemma 4.1.

For (ii), recall the notation $\tilde{A} = (A, j(A, \tau))$ for each $A \in \mathrm{SL}_2(\mathbb{Z})$. Define the 2-cocycle $\sigma(\cdot, \cdot)$ of $\mathrm{SL}_2(\mathbb{Z})$ by

$$\sigma(A, B) = j(A, B\tau)j(B, \tau)j(AB, \tau)^{-1},$$

which takes values in $\{\pm 1\}$ and $(1, \sigma(A, B))\widetilde{AB} = \tilde{A}\tilde{B}$ for $A, B \in \mathrm{SL}_2(\mathbb{Z})$. For explicit formulas of σ , see Theorem 4.1 of [19]. Now note that $\rho_D(Z)\mathbf{e}_\gamma = e(-\frac{r}{4})\mathbf{e}_{-\gamma}$, so we have $\rho_D(Z^2)\mathbf{e}_\gamma = -\mathbf{e}_\gamma$, for any γ . Let $B \in \Gamma_0(N)$ and $A \in \mathrm{SL}_2(\mathbb{Z})$. We have

$$\begin{aligned} & \left(f|_{\tau_N \widetilde{BA}} \right) \rho_D(\widetilde{BA})^{-1} \mathbf{e}_0 = \sigma(B, A)^2 \left(f|_{\tau_N \tilde{B}\tilde{A}} \right) \rho_D(\tilde{B}\tilde{A})^{-1} \mathbf{e}_0 \\ & = \nu(B)^{2k} \left(f|_{\tau_N B^* \tilde{A}} \right) \rho_D(\tilde{A})^{-1} \rho_D(\tilde{B})^{-1} \mathbf{e}_0 = \left(f|_{\tau_N \tilde{A}} \right) \rho_D(\tilde{A})^{-1} \mathbf{e}_0, \end{aligned}$$

since $f|_{\tau_N} \in \mathcal{A}(N, k, \chi)$ by Proposition 1.4 of [18]. It follows that the sum is independent of the choice of representatives. Now for any $B \in \mathrm{SL}_2(\mathbb{Z})$,

$$\begin{aligned} F|_{\tilde{B}} &= \sum_{A \in \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z})} \left(f|_{\tau_N \tilde{A}\tilde{B}} \right) \rho_D(\tilde{A})^{-1} \mathbf{e}_0 \\ &= \sum_{A \in \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z})} \sigma(A, B) \left(f|_{\tau_N \widetilde{AB}} \right) \rho_D(\tilde{B}) \rho_D(\tilde{A}\tilde{B})^{-1} \mathbf{e}_0 \\ &= \sigma(A, B)^2 \rho_D(\tilde{B}) F = \rho_D(\tilde{B}) F. \end{aligned}$$

Moreover, it is clear that

$$F|_{Z^2} = -F = \rho_D(Z^2)F,$$

and since every element in $\mathrm{Mp}_2(\mathbb{Z})$ is equal to \tilde{B} or $Z^2\tilde{B}$ for some $B \in \mathrm{SL}_2(\mathbb{Z})$, we have $F \in \mathcal{A}(k, \rho_D)$. Finally, the actions of $\mathrm{Mp}_2(\mathbb{Z})$ and $\mathrm{Aut}(D)$ commute, so we see easily that F is $\mathrm{Aut}(D)$ -invariant, finishing the proof. \square

We define a quantity that will appear in the isomorphism below. For each integer n , define

$$(6) \quad s(n) = s_D(n) = \prod_{p:p|(M,n)} \left(1 + \frac{p}{|D_p|} \right).$$

It is clear that $s(n) = \sum_{m|(M,n)} \frac{m}{|D_m|}$, so in particular $s(0) = \sum_{m|M} \frac{m}{|D_m|}$. Set $n' = \frac{M}{(M,n)}$, and it is also clear that

$$(7) \quad s(n)s(n') = s(0).$$

If $\frac{n}{N}$ is a norm, we have the formula

$$(8) \quad \#\left\{\gamma \in D: Q(\gamma) = \frac{n}{N}\right\} = s(n') \frac{|D_{n'}|}{n'}.$$

Indeed, it suffices to prove it on each local component by the Chinese Remainder Theorem, namely we only need to consider $n = N/4$ or $n = N/p$ for odd prime divisor p of N . For D_2 both sides are 1 and for D_p it follows from Lemma 2.2.

By Lemma 5.1, we may define linear maps $\phi_D: \mathcal{A}^{\text{inv}}(k, \rho_D) \rightarrow \mathcal{A}(N, k, \chi')$ by

$$F \mapsto i^{\frac{2k-r}{2}} s(0)^{-1} |D|^{\frac{1}{2}} N^{-\frac{k}{2}} F_0 |_{\tau_N},$$

and $\psi_D: \mathcal{A}(N, k, \chi') \rightarrow \mathcal{A}^{\text{inv}}(k, \rho_D)$ by

$$f \mapsto i^{\frac{2k-r}{2}} 3^{-1} |D|^{-\frac{1}{2}} N^{\frac{k}{2}} \sum_{A \in \Gamma_0(N) \backslash \text{SL}_2(\mathbb{Z})} \left(f |_{\tau_N \tilde{A}} \right) \rho_D(\tilde{A})^{-1} \mathbf{e}_0.$$

If there is no danger of confusion, we shall drop the subscript D and write ϕ and ψ .

We first prove one side of the isomorphism. For two cusps $s, s' \in \mathbb{P}^1(\mathbb{Q})$, we write $s \sim s'$ if they are equivalent modulo $\Gamma_0(N)$.

Lemma 5.2. We have $\psi \circ \phi = \text{id}$.

Proof. By Lemma 2.3 (i), we need to prove that for $F \in \mathcal{A}^{\text{inv}}(k, \rho_D)$,

$$\sum_{A \in \Gamma_0(N) \backslash \text{SL}_2(\mathbb{Z})} F_0 |_{\tilde{A}} \langle \rho_D(\tilde{A})^{-1} \mathbf{e}_0, \mathbf{e}_0 \rangle = 3s(0)F_0.$$

Let s be any cusp of $\Gamma_0(N)$ and consider the sub-sum

$$F_s = \sum_{A: A\infty \sim s} F_0 |_{\tilde{A}} \langle \rho_D(\tilde{A})^{-1} \mathbf{e}_0, \mathbf{e}_0 \rangle,$$

where the sum is over all cosets $A \in \Gamma_0(N) \backslash \text{SL}_2(\mathbb{Z})$ such that the cusp $A\infty$ is equivalent to s . It is clear that F_s is \tilde{T} -invariant. If $s \sim \frac{1}{N/m}$ is a cusp with $m | M$, it is well-known that the coset representatives which map to the cusp s can be chosen of the form $\gamma_m T^j$, $j \bmod m$, so

$$\begin{aligned} F_s &= \sum_{j \bmod m} F_0 |_{\widetilde{\gamma_m T^j}} \langle \rho_D(\widetilde{\gamma_m})^{-1} \mathbf{e}_0, \rho_D(\widetilde{T^j}) \mathbf{e}_0 \rangle \\ &= \sum_{j \bmod m} \sum_{\alpha \in D} F_\alpha \langle \rho_D(\widetilde{\gamma_m T^j}) \mathbf{e}_\alpha, \mathbf{e}_0 \rangle \langle \rho_D(\widetilde{\gamma_m})^{-1} \mathbf{e}_0, \mathbf{e}_0 \rangle \\ &= \sum_{j \bmod m} F_0 \langle \rho_D(\widetilde{\gamma_m T^j}) \mathbf{e}_0, \mathbf{e}_0 \rangle \langle \rho_D(\widetilde{\gamma_m})^{-1} \mathbf{e}_0, \mathbf{e}_0 \rangle \\ &= \sum_{j \bmod m} F_0 \langle \rho_D(\widetilde{\gamma_m}) \mathbf{e}_0, \mathbf{e}_0 \rangle \langle \rho_D(\widetilde{\gamma_m})^{-1} \mathbf{e}_0, \mathbf{e}_0 \rangle = \frac{m}{|D_m|} F_0, \end{aligned}$$

where we applied Lemma 2.3 (ii) in the third equality and Theorem 6.4 of [19] to compute $|\langle \rho_D(\widetilde{\gamma}_m)\mathbf{e}_0, \mathbf{e}_0 \rangle|^2 = \frac{1}{|D_m|}$ in the last equality. If $s \sim \frac{1}{M/m}$, then by the same computation with γ_{4m} in place of γ_m , we have

$$F_s = \sum_{j \bmod 4m} F_0 \langle \rho_D(\widetilde{\gamma}_{4m})\mathbf{e}_0, \mathbf{e}_0 \rangle \langle \rho_D(\widetilde{\gamma}_{4m})^{-1}\mathbf{e}_0, \mathbf{e}_0 \rangle = 4m \cdot \frac{1}{2|D_m|} F_0 = \frac{2m}{|D_m|} F_0.$$

The cusps $s \sim \frac{1}{2M/m}$ give $F_s = 0$, since matrices γ sending ∞ to s give non-trivial x_c in Theorem 6.4 of [19]. The above cases cover exactly all of the cusps of $\Gamma_0(N)$ and the lemma follows. \square

We also use ψ to denote the restriction of ψ to the subspace $\mathcal{A}^\epsilon(N, k, \chi')$. Now we prove that on the subspaces we constructed in the preceding section, ϕ and ψ are isomorphisms.

Theorem 5.3. *Let D be anisotropic with $D_2 = 2_t^{+1}$ and let (N, χ', ϵ) correspond to D . The maps ϕ and ψ are inverse isomorphisms between $\mathcal{A}^{inv}(k, \rho_D)$ and $\mathcal{A}^\epsilon(N, k, \chi')$. Explicitly, if $f = \sum_n a(n)q^n \in \mathcal{A}^\epsilon(N, k, \chi')$ and $\psi(f) = F = \sum_\gamma F_\gamma \mathbf{e}_\gamma$, then*

$$F_\gamma(\tau) = \sum_{n \equiv NQ(\gamma) \bmod NZ} s(n) \frac{M/(M, n)}{|D_{M/(M, n)}|} a(n) q^{\frac{n}{N}}.$$

Proof. By Lemma 5.1, $\psi(f)$ is $\text{Aut}(D)$ -invariant and we need to show that $\phi(F)$ satisfies the ϵ -condition. Since

$$\tau_N = \tilde{S} \left(\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}, N^{-\frac{1}{4}}(-i)^{\frac{1}{2}} \right)$$

and

$$F|\tilde{S} = \rho_D(\tilde{S})F = \sum_{\gamma \in D} F_\gamma \rho_D(\tilde{S})\mathbf{e}_\gamma = i^{-\frac{r}{2}} |D|^{-\frac{1}{2}} \sum_{\gamma \in D} F_\gamma \sum_{\delta \in D} e(-\langle \gamma, \delta \rangle) \mathbf{e}_\delta,$$

we have

$$\begin{aligned} \phi(F) &= i^{\frac{2k-r}{2}} s(0)^{-1} |D|^{\frac{1}{2}} N^{-\frac{k}{2}} \langle F|\tau_N, \mathbf{e}_0 \rangle \\ &= i^{\frac{2k-r}{2}} s(0)^{-1} |D|^{\frac{1}{2}} N^{-\frac{k}{2}} N^{\frac{2k}{4}} (-i)^{-\frac{2k}{2}} \langle (F|\tilde{S})(N\tau), \mathbf{e}_0 \rangle \\ &= i^{\frac{2k-r}{2}} s(0)^{-1} |D|^{\frac{1}{2}} i^{\frac{2k}{2}} i^{-\frac{r}{2}} |D|^{-\frac{1}{2}} \sum_{\gamma \in D} F_\gamma(N\tau), \end{aligned}$$

which simplifies to

$$(9) \quad \phi(F) = s(0)^{-1} \sum_{\gamma \in D} F_\gamma(N\tau) := \sum_n c(n)q^n.$$

Now by the action of T , the Fourier coefficient $a_\gamma(n)$ of $F_\gamma(N\tau)$ vanishes unless $\frac{n}{N} \equiv Q(\gamma)$ for some γ . It follows that $c(n) = 0$ unless $\frac{n}{N}$ is a norm, and by Lemma 4.3, this means that $\phi(F)$ satisfies the desired ϵ_p -condition for $p = 2$ and for $p \mid M$ with $\chi_p \neq 1$.

When $p \mid M$ with $\chi_p = 1$, we need to show that $\phi(F)|Y(p) = -\phi(F)$, that is, $F_0|\tau_N Y(p) = -F_0|\tau_N$. By Proposition 3.1 (i) and (v), we only have to show that

$$\sum_{j \bmod p} F_0|\gamma_p^* \tilde{T}^j = -F_0.$$

Indeed,

$$\begin{aligned}
& \sum_{j \bmod p} F_0 | \gamma_p^* \tilde{T}^j = \nu(\gamma_p)^{-2k} \sum_{j \bmod p} F_0 | \tilde{\gamma}_p \tilde{T}^j = \nu(\gamma_p)^{-2k} \sum_{j \bmod p} \langle F, \mathbf{e}_0 \rangle | \tilde{\gamma}_p \tilde{T}^j \\
& = \nu(\gamma_p)^{-2k} \sum_{j \bmod p} \langle F | \tilde{\gamma}_p, \mathbf{e}_0 \rangle | \tilde{T}^j = \nu(\gamma_p)^{-2k} \sum_{j \bmod p} \langle F, \rho_D(\tilde{\gamma}_p)^{-1} \mathbf{e}_0 \rangle | \tilde{T}^j \\
& = \nu(\gamma_p)^{-2k} \sum_{\alpha \in D_p} \sum_{j \bmod p} \overline{\xi(\tilde{\gamma}_p)} p^{-1} F_\alpha | \tilde{T}^j,
\end{aligned}$$

where we applied Theorem 6.4 of [19] and $\xi(\tilde{\gamma}_p) = \xi(a, c)$ with $\gamma_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By Lemma 2.3 (ii) this is equal to $\nu(\gamma_p)^{-2k} \overline{\xi(\tilde{\gamma}_p)} F_0$. We now apply the explicit formulas of [19] in terms of notations therein to compute $\xi(\tilde{\gamma}_p)$. Since by our construction γ_p satisfies $c > 0$ and

$$(10) \quad a, d \equiv 1 \pmod{16(M/p)^2}, \quad a, d \equiv 0 \pmod{p^2}, \quad c \equiv 0 \pmod{16(M/p)^2}, \quad c \equiv 1 \pmod{p^2},$$

it is straightforward to obtain that

$$\xi_0 = \left(\frac{-a}{c} \right), \quad \xi_2 = e_8(-(c_2 + 1 + t)(a + 1)) = e_4(-(c_2 + 1 + t)) = - \left(\frac{-1}{c_2} \right) e_4(-t)$$

(note that there is a typo in the definition of ξ_2 on page 522 of [19] where $c_2 - 1$ should be $c_2 + 1$). It follows that $\xi_0 \xi_2 = - \left(\frac{a}{c} \right) e_4(-t)$. Moreover, For $J = p^{\pm 2}$, $\xi(J) = -1$, while for $J = p'^{\pm 2}$ with odd prime $p' \neq p$, $\xi(J) = 1$. For $J = p'^{\pm 1}$, $\xi(J) = \left(\frac{-a}{p'} \right) = \left(\frac{-1}{p'} \right) = \chi_{p'}(-1)$. Finally, for $J = 2_t^{\pm 1}$, $\xi(J) = 1$. Putting everything together, we have

$$\xi(\tilde{\gamma}_p) = e_4(-r - t) \left(\frac{a}{c} \right) \chi_M(-1) = e_4(-r - t) e_4(r - t) \left(\frac{a}{c} \right) = - \left(\frac{a}{c} \right) = - \left(\frac{d}{c} \right) = - \left(\frac{c}{d} \right),$$

where we applied (4) in the second equality and the quadratic reciprocity law in the last equation. The congruences of (10) were involved in all of such computations and they also imply that $\nu(\gamma_p) = \left(\frac{c}{d} \right)$. Therefore, we have $\nu(\gamma_p)^{-2k} \overline{\xi(\tilde{\gamma}_p)} = -1$ as desired.

By Lemma 5.2, it remains to prove that $\phi \circ \psi = id$. Let $f \in \mathcal{A}^\epsilon(N, k, \chi')$. For each cusp s , we modify F_s as

$$F'_s = \sum_{A: A\infty \sim s} \left(f | \tau_N \tilde{A} \tau_N \right) \langle \rho_D(\tilde{A})^{-1} \mathbf{e}_0, \mathbf{e}_0 \rangle,$$

and we need to show that $\sum_s F'_s = 3s(0)f$. The computations for $s \sim \frac{1}{N/m}$ and for $s \sim \frac{1}{M/m}$ with $m \mid M$ are similar and all other cusps give 0. We only have to show that

$$F'_s = \frac{m}{|D_m|} f \quad \text{and} \quad F'_s = \frac{2m}{|D_m|} f$$

respectively in the two cases. Since the former is similar but easier, we omit it and only treat the case $s \sim \frac{1}{M/m}$. In this case, we have

$$\begin{aligned}
F'_s &= \sum_{j \bmod 4m} f | \tau_N \widetilde{\gamma_{4m}} T^j \tau_N \langle \rho_D(\widetilde{\gamma_{4m}})^{-1} \mathbf{e}_0, \rho_D(\tilde{T}^j) \mathbf{e}_0 \rangle \\
&= \sum_{j \bmod 4m} f | \tau_N \tilde{S} \widetilde{\gamma_{M/m}}^{-1} \tilde{T}^j \tau_N \langle \rho_D(\widetilde{\gamma_{4m}})^{-1} \mathbf{e}_0, \mathbf{e}_0 \rangle \\
&= e_8(-2k) \sum_{j \bmod 4m} f | \widetilde{\delta_N}^{-1} \widetilde{\gamma_{M/m}}^{-1} \tilde{T}^j \tau_N \langle \rho_D(\widetilde{\gamma_{4m}})^{-1} \mathbf{e}_0, \mathbf{e}_0 \rangle,
\end{aligned}$$

where $e_8(-2k)$ comes from the extra factor $(1, \sqrt{-i})$ of τ_N in (3) and the relevant σ -factor is trivial. Assume that

$$\gamma_{M/m} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \beta = \begin{pmatrix} dm/M & -b/4m \\ -4mc & aM/m \end{pmatrix}.$$

Then it is easy to see that $\beta \in \Gamma_0(4)$,

$$(11) \quad N \widetilde{\delta}_N^{-1} \widetilde{\gamma}_{M/m}^{-1} \widetilde{\delta}_{4m} = 4m \widetilde{\beta} \widetilde{\delta}_{M/m},$$

and

$$(12) \quad \beta \gamma_{M/m}^{-1} \equiv \begin{cases} \begin{pmatrix} (M/m)^{-1} & 0 \\ 0 & M/m \end{pmatrix} & \text{mod } 4m, \\ \begin{pmatrix} -(4m)^{-1} & 0 \\ 0 & -4m \end{pmatrix} & \text{mod } M/m. \end{cases}$$

In particular $\beta \gamma_{M/m}^{-1} \in \Gamma_0(N)$. Observe that $\sigma(\beta \gamma_{M/m}^{-1}, \gamma_{M/m}) = 1$ by the assumption that all of the entries of $\gamma_{M/m}$ are positive. By (11), we obtain

$$\begin{aligned} F'_s &= e_8(-2k) \sum_{j \bmod 4m} f | \widetilde{\beta} \widetilde{\delta}_{M/m} \widetilde{\delta}_{4m}^{-1} \widetilde{T}^j \tau_N \langle \rho_D(\widetilde{\gamma}_{4m})^{-1} \mathbf{e}_0, \mathbf{e}_0 \rangle \\ &= e_8(-2k) \sum_{j \bmod 4m} f | \beta \gamma_{M/m}^{-1} \widetilde{\gamma}_{M/m} \widetilde{\delta}_{M/m} \widetilde{\delta}_{4m}^{-1} \widetilde{T}^j \tau_N \langle \rho_D(\widetilde{\gamma}_{4m})^{-1} \mathbf{e}_0, \mathbf{e}_0 \rangle \\ &= e_8(-2k) \nu(\beta \gamma_{M/m}^{-1})^{2k} \chi_{4m}(M/m) \chi_{M/m}(-4m) \\ &\quad \times \sum_{j \bmod 4m} f | \widetilde{\gamma}_{M/m} \widetilde{\delta}_{M/m} \widetilde{\delta}_{4m}^{-1} \widetilde{T}^j \tau_N \langle \rho_D(\widetilde{\gamma}_{4m})^{-1} \mathbf{e}_0, \mathbf{e}_0 \rangle, \end{aligned}$$

where the ν -factor comes from the difference of \tilde{A} and A^* (and we employ χ instead of χ') and the two character values appear because of (12). It further simplifies to

$$(13) \quad \begin{aligned} F'_s &= (4m)^{1-\frac{k}{2}} e_8(-2k) \nu(\beta)^{2k} \chi_{4m}(M/m) \chi_{M/m}(-4m) \\ &\quad \times f | W(M/m) U(4m) \tau_N \langle \rho_D(\widetilde{\gamma}_{4m})^{-1} \mathbf{e}_0, \mathbf{e}_0 \rangle. \end{aligned}$$

Applying Proposition 3.1 in the order of (iv), (iii), (v), (iii) and then the ϵ -condition of f , we see that

$$(14) \quad \begin{aligned} (4m)^{1-\frac{k}{2}} F_0 | W(M/m) U(4m) \tau_N &= \varepsilon_m^{2k} 2^{\frac{3}{2}} \left(\chi_2(-1) \left(\frac{-1}{M} \right) \right)^{k+1} (-1)^{\lfloor \frac{2k+1}{4} \rfloor} \chi_m(-1) \chi_{N/m}(m) \\ &\quad \times \left(\frac{M/m}{m} \right) \left(\prod_{p|m, \chi_p=1} (-1) \right) \left(\prod_{p|m, \chi_p \neq 1} \epsilon_p \varepsilon_p^{-1} p^{\frac{1}{2}} \chi_p(m/p) \right) f. \end{aligned}$$

In the lengthy computation of (14), one should keep track of the correct characters. It is easy to see that

$$(15) \quad \nu(\beta)^{2k} = \left(\frac{-4mc}{aM/m} \right) \varepsilon_{M/m}^{-2k} = \left(\frac{c}{d} \right) \left(\frac{m}{M/m} \right) \varepsilon_{M/m}^{2k}.$$

Finally, by Theorem 6.4 of [19], we have

$$(16) \quad \langle \rho_D(\widetilde{\gamma_{4m}})^{-1} \mathbf{e}_0, \mathbf{e}_0 \rangle = |D_{4m}|^{-\frac{1}{2}} e_8(2r-t) \chi_{M/m}(-1) \left(\frac{c}{d}\right) \\ \times \left(\prod_{p|m, \chi_p=1} (-1) \right) \left(\prod_{p|m, \chi_p \neq 1} \epsilon_p \epsilon_p^{-1} \chi_p(N/p) \right).$$

Let $\eta = -1$ if $\chi_2(-1) = -1$ and $\left(\frac{-1}{M}\right) = -1$ and $\eta = 1$ otherwise. Plugging (14), (15) and (16) into (13), we have

$$F'_s = \frac{2m}{|D_m|} e_8(t-2k) \eta \chi_2(M) \chi_2(-1)^k \left(\frac{2k}{2}\right) f.$$

Now by (4) and that $2k \equiv r \pmod{4}$, we have $\chi_2(-1) = 1$ if and only if $2k \equiv t \pmod{4}$. By applying the formula (5.6) of [19] and checking cases, we have $F'_s = \frac{2m}{|D_m|} f$ as desired.

Finally, the explicit formula follows easily from (9), together with the formulas (7) and (8). This finishes the proof. \square

Corollary 5.4. Let f^ϵ be the ϵ -component of $f \in \mathcal{A}(N, k, \chi')$ with respect to the decomposition into common eigen-subspaces. Then $\psi_D(f) = 0$ if and only if $f^\epsilon = 0$.

Proof. First assume that $f \in \mathcal{A}(N, k, \chi')$ is such that $f|Y(p) = pf$ for an odd $p \mid M$ and let $s \sim \frac{1}{N/m}$ and $s' \sim \frac{1}{N/pm}$ or $s \sim \frac{1}{M/m}$ and $s' \sim \frac{1}{M/pm}$ for $m \mid M/p$. In the proof of Theorem 5.3, by applying Proposition 3.1 to separate and shift the $W(p)$ operator (without isolating other primes) in (13), we can obtain that $F'_s = -F'_{s'}$, hence $\sum_s F'_s = 0$. It follows that $\phi(\psi(f)) = 0$, but $\psi(f) \in \mathcal{A}^{\text{inv}}(k, \rho_D)$ and ϕ is injective, so $\psi(f) = 0$. The case when $f|Y(p) = -\epsilon_p \epsilon_p^{-1} p^{\frac{1}{2}} f$, is similar. Finally, if $f|Y(4) = \mu_2^- f$, then $F'_s = -F'_{s'}$ where $s \sim \frac{1}{M/m}$ and $s' \sim \frac{1}{N/m}$, so we also have $\psi(f) = 0$. In other words, we always have $\psi(f) = \psi(f^\epsilon)$ for $f \in \mathcal{A}(N, k, \chi')$, so the corollary follows from Theorem 5.3. \square

Corollary 5.5. The isomorphisms ϕ and ψ induces isomorphisms

$$\mathcal{M}^{\text{inv}}(k, \rho_D) \simeq \mathcal{M}^\epsilon(N, k, \chi') \quad \text{and} \quad \mathcal{S}^{\text{inv}}(k, \rho_D) \simeq \mathcal{S}^\epsilon(N, k, \chi').$$

Consequently, $f \in \mathcal{A}^\epsilon(N, k, \chi')$ is holomorphic or cuspidal if and only if f is holomorphic or vanishes at ∞ , respectively.

Proof. Let $F = \sum_\gamma F_\gamma \mathbf{e}_\gamma \in \mathcal{A}^{\text{inv}}(k, \rho_D)$. Since for each $A \in \text{SL}_2(\mathbb{Z})$, $F|\tilde{A} = \sum_\gamma F_\gamma \rho_D(\tilde{A}) \mathbf{e}_\gamma$, F is holomorphic or cuspidal if and only if F_γ is holomorphic or vanishes at ∞ for each γ , respectively. By Lemma 2.3, this is equivalent to saying that $F_0|\tilde{A}$ is holomorphic or vanishes at ∞ respectively for all $A \in \text{SL}_2(\mathbb{Z})$, that is F_0 is holomorphic or cuspidal, respectively. This in turn is equivalent to saying that $\phi(F)$ is holomorphic or cuspidal since τ_N preserves the holomorphic subspace and the cuspidal subspace. \square

The proof of Theorem 5.3 actually indicates that the same result holds if we relax the condition of holomorphy, but we only state it in the special case where $\chi' = 1$, which will be needed in Section 6.

Corollary 5.6. Let D correspond to $(N, 1, \epsilon)$. Let $f = \sum_n a_n(y)e(nx)$ be a real analytic modular form of level N , weight k and trivial character that satisfies the ϵ -condition. Then $f \mapsto F = \sum_\gamma F_\gamma \mathbf{e}_\gamma$ with

$$F_\gamma(\tau) = \sum_{n \equiv NQ(\gamma) \pmod{N\mathbb{Z}}} s(n)a_n(y/N)e(nx/N)$$

defines an isomorphism to the space of $\text{Aut}(D)$ -invariant modular forms of weight k and type ρ_D .

Proof. The proofs above for $\phi \circ \psi = \text{id}$ and $\psi \circ \phi = \text{id}$ and that of Lemma 2.3 are independent of whether f or F are holomorphic. The assumption implies that $m = |D_m|$ for each $m \mid M$ and the corollary follows. \square

6. ZAGIER DUALITY AND BORCHERDS' THEOREM

As in the preceding section, let D be anisotropic with $D_2 = 2_t^{+1}$, so $N = 4M$ with M being odd and squarefree. For the rest of this paper, we assume further that $\chi_p \neq 1$ for each $p \mid M$, so $\chi' = 1$ and D is cyclic of order $2M$. Such discriminant forms are considered, for example, in [2] and [5].

We extend the notion of reduced modular forms in [25] to the current setting: for any integer m , $f = \sum_n a(n)q^n \in \mathcal{A}^\epsilon(N, k, 1)$ is called reduced of order m if $f = \frac{1}{s(m)}q^m + O(q^{m+1})$ and if for each $n > m$ with $a(n) \neq 0$, there does not exist $g \in \mathcal{A}^\epsilon(N, k, 1)$ such that $g = q^n + O(q^{n+1})$. A reduced modular form of order m , if exists, must be unique and we denote it by f_m . The set of reduced modular forms is clearly a basis for $\mathcal{A}^\epsilon(N, k, 1)$. Essentially, up to the leading scalars $\frac{1}{s(m)}$, the basis of reduced modular forms can be viewed as the reduced row echlon form obtained from any given basis.

We first consider the existence of f_m for $m < 0$. Let D^* be the dual discriminant form of D given by the same abelian group with the quadratic form $-Q(\cdot)$. It is clear that D^* is also anisotropic and the corresponding data is (N, χ', ϵ^*) with $\chi' = 1$, $\epsilon_2^* = -\epsilon_2$ and $\epsilon_p^* = \chi_p(-1)\epsilon_p$ if $p \mid M$. We denote the reduced modular forms in $\mathcal{A}^{\epsilon^*}(N, 2 - k, 1)$ by f_m^* .

Proposition 6.1. Let $B^* = \{m: f_m^* \in \mathcal{M}^{\epsilon^*}(N, 2 - k, 1) \text{ exists}\}$. Then for any $m < 0$ with $\chi_p(m) \neq -\epsilon_p$ for all $p \mid M$, we have that $f_m \in \mathcal{A}^\epsilon(N, k, 1)$ exists if and only if $-m \notin B^*$.

Proof. The obstruction theorem, Theorem 3.1 of [3] implies the following: let $P = \sum_{n \leq 0} a(n)q^n$ be a polynomial in q^{-1} with $a(n) = 0$ if $\chi_p(n) = -\epsilon_p$ for some $p \mid M$ or $n \equiv 2, -\epsilon_2 \pmod{4}$. Then there exists $f \in \mathcal{A}^\epsilon(N, k, 1)$ with $f = \sum_n a(n)q^n$ if and only if

$$\sum_{n \leq 0} s(n)a(n)b(-n) = 0$$

for each $g = \sum_n b(n)q^n \in \mathcal{M}^{\epsilon^*}(N, 2 - k, 1)$. So in general, a reduced modular form does not always contain one negative exponent unless $\mathcal{M}^{\epsilon^*}(N, 2 - k, 1)$ is trivial (see Example 5.10 in [25] in the case of integral weight).

If $-m \in B^*$, consider $g = f_{-m}^*$ and we see that $a(n) = 0$ if $n < m$ and $b(-n) = 0$ if $n > m$. Therefore, the linear equation of obstruction given by f_{-m}^* is $s(-m)a(m)b(-m) = 0$, so $a(m) = 0$

and f_m does not exist. Conversely, if $-m \notin B^*$, write $B^* = \{n_i : i\}$ and $f_{n_i}^* = \sum_n b_i(n)q^n$, so that $s(n_i)b_j(n_i) = \delta_{ij}$. Let

$$P = \frac{1}{s(m)}q^m - \sum_{i: -n_i > m} b_i(-m)q^{-n_i},$$

and we check that P satisfies the linear system of obstructions, from which the existence of f_m follows. For each n_j , the linear equation from the obstruction of $f_{n_j}^*$ reads

$$(17) \quad b_j(-m) - \sum_{i: -n_i > m} s(-n_i)b_i(-m)b_j(n_i) = 0.$$

If $n_j < -m$, then the first term of the left-hand side of (17) is zero and the sum is also zero since n_j does not appear and $s(n_i)b_j(n_i) = \delta_{ij}$. On the other hand, if $n_j > -m$, since $s(n_i)b_j(n_i) = \delta_{ij}$ and $s(-n) = s(n)$, the left-hand side of (17) is equal to $b_j(-m) - b_j(-m) = 0$ as desired. \square

Now we prove the Zagier duality.

Theorem 6.2. *Let m, d be integers and assume that both of the reduced modular forms*

$$f_m = \sum_n a_m(n)q^n \in \mathcal{A}^\epsilon(N, k, 1)$$

$$f_d^* = \sum_n a_d^*(n)q^n \in \mathcal{A}^{\epsilon^*}(N, 2 - k, 1)$$

exist. Then $a_m(-d) = -a_d^(-m)$.*

Proof. The statement is trivial when $m + d > 0$, so by symmetry we may assume that $m = d = 0$ or $m < 0$.

Let $F = \psi(f_m)$ and $G = \psi(f_d^*)$. It is clear that $H = \sum_\gamma F_\gamma G_\gamma$ is a weakly holomorphic modular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$. Therefore, the sum of the residues of the meromorphic 1-form $H(\tau)d\tau$ on the compact Riemann surface $X(1)$ vanishes. F and G are holomorphic on \mathbb{H} and the residue of $H(\tau)d\tau$ at ∞ is given by $\frac{s(0)}{2\pi i}$ times $\sum_{n \in \mathbb{Z}} s(n)a_m(n)a_d^*(-n)$. If $(m, d) = (0, 0)$, we have $s(0)a_m(0)a_d^*(0) = 0$, contradicting to the existence of f_0 and f_0^* . We then assume $m < 0$, we have

$$\sum_{n \in \mathbb{Z}} s(n)a_m(n)a_d^*(-n) = a_m(-d) + a_d^*(-m) + \sum_{m < n < -d} s(n)a_m(n)a_d^*(-n) = 0.$$

So we only have to prove that $\sum_{m < n < -d} s(n)a_m(n)a_d^*(-n) = 0$. If $m < n \leq 0$, then $a_m(n) = 0$ if $-n \in B^*$ and $a_d^*(-n) = 0$ if $-n \notin B^*$. Similarly, if $0 < n < -d$ and $a_d^*(-n) \neq 0$, then $-n \notin B^*$ and f_n exists, and from the definition of reduced modular forms, we have $a_m(n) = 0$. \square

In the rest of this section, we write down explicitly the Borcherds lift in the case of $\mathrm{O}(2, 1)$. The following even lattice

$$L = \left\{ \begin{pmatrix} a & b/M \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$$

with $Q(\alpha) = -M \det(\alpha)$ and $(\alpha, \beta) = M \operatorname{tr}(\alpha\beta)$ was considered in [5]. We know that $D = L'/L \simeq \mathbb{Z}/2M\mathbb{Z}$ with $D = \prod_{p|2M} D_p$:

$$D_2 = 2_t^{+1}, t = \left(\frac{-1}{M}\right), \quad D_p = p^{\delta_p}, \delta_p = \left(\frac{2M/p}{p}\right), p | M.$$

It follows that for such D , $\epsilon_2 = +1$ and $\epsilon_p = +1$ for all $p | M$ and $\chi' = \chi\left(\frac{N}{\cdot}\right) = 1$, so we shall simply denote $\mathcal{A}^+(N, k, 1)$ for $\mathcal{A}^\epsilon(N, k, \chi')$. The dual D^* of D then gives ϵ^* with $\epsilon_2^* = -1$ and $\epsilon_p^* = \chi_p(-1)$ for $p | M$.

Let us recall Zagier's non-holomorphic modular form

$$\mathbf{G}(\tau) = \sum_{n=0}^{\infty} H(n)q^n + \frac{1}{16\pi} \sum_{n \in \mathbb{Z}} q^{-n^2} \int_y^{\infty} e^{-4\pi un^2} u^{-\frac{3}{2}} du$$

for $\Gamma_0(4)$ of weight $\frac{3}{2}$ (see [22]). Here $H(n)$ denotes the Hurwitz class number of n , whose generating function will be denoted by $G(\tau) = \sum_{n=0}^{\infty} H(n)q^n$. Consider \mathbf{G} as of level N and denote by \mathbf{G}^* the ϵ^* -component of \mathbf{G} (that is, $\mathbf{G}^*|Z(p) = \epsilon_p^* \mathbf{G}^*$ and \mathbf{G}^* is in Kohnen's plus space) and denote its holomorphic part by $G^*(\tau) = \sum_{n=0}^{\infty} H^*(n)q^n$. The non-holomorphic part of \mathbf{G} clearly satisfies the ϵ^* -condition, so \mathbf{G} and \mathbf{G}^* share the same non-holomorphic part and the difference is holomorphic. The map ψ_D in the preceding section can be extended to non-holomorphic modular forms, so $\psi_{D^*}(\mathbf{G})$ is a non-holomorphic modular form of weight $\frac{3}{2}$ and type ρ_{D^*} by Corollary 5.6 and we denote it by $\mathbf{G}_N(\tau) = \psi_{D^*}(\mathbf{G})$. By Corollary 3.3 and the fact that $H(n) \neq 0$ for any positive $n \equiv 0, 3 \pmod{4}$, we see that $\mathbf{G}_N(\tau)$ is non-zero. We denote the holomorphic part of $\mathbf{G}_N(\tau)$ by $G_N(\tau)$.

Now we can apply our results to evaluate the Borcherds lift of $\psi_D(f)$ in Theorem 13.3 of [2]:

Theorem 6.3. *Let D and $(N, 1, \epsilon)$ correspond as above. Assume $f = \sum_n c(n)q^n \in \mathcal{A}^+(N, k, 1)$ with $s(n)c(n) \in \mathbb{Z}$ for all $n \leq 0$. Then there exists a meromorphic modular form $\Psi(f)$ of weight $s(0)c(0)$ for $\Gamma_0(M)$ (with some finite multiplier system) such that*

(1) $\Psi(f)$ has an infinite product expression:

$$\Psi(f)(\tau) = q^\rho \prod_{n=1}^{\infty} (1 - q^n)^{s(n^2)c(n^2)},$$

where $\rho = -\sum_{n \in \mathbb{Z}} s(n)c(-n)H^*(n)$.

(2) The zeros and poles of $\Psi(f)$ on \mathbb{H} occur at CM points τ of (negative) discriminant d with order

$$\sum_{n=1}^{\infty} s(dn^2)c(dn^2).$$

Proof. In Theorem 13.3 of [2],

$$z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad z' = \begin{pmatrix} 0 & 1/M \\ 0 & 0 \end{pmatrix}, \quad K \simeq \mathbb{Z} \text{ with } Q(n) = n^2.$$

Let $F = \psi_D(f) = \sum_{\gamma} F_{\gamma} \mathbf{e}_{\gamma}$ and by Corollary 5.6,

$$F_{\gamma}(\tau) = \sum_{n \equiv NQ(\gamma) \pmod{NZ}} s(n)c(n)q^{\frac{n}{N}}.$$

By Theorem 13.3 of [2], the Borcherds lift of F is equal to

$$\Psi(f) := \Psi(F) = q^{\rho} \prod_{n=1}^{\infty} (1 - q^n)^{s(n^2)c(n^2)},$$

where ρ comes from the Weyl vector.

For the formula of ρ , let Θ_{2M} be as in Lemma 9.5 of [2] and denote

$$\Theta_{2M}^{\text{inv}} = |\text{Aut}(D)|^{-1} \sum_{\sigma \in \text{Aut}(D)} \Theta_{2M}^{\sigma}.$$

By comparing the \mathbf{e}_0 -components and by Lemma 2.3, we see that $\Theta_{2M}^{\text{inv}} = s(0)^{-1} \psi_D(\theta)$, where θ is the usual level 4 theta function. Since F is $\text{Aut}(D)$ -invariant, Corollary 9.6 of [2] will give the same value if we replace Θ_{2M} by Θ_{2M}^{σ} or by Θ_{2M}^{inv} . It follows easily that ρ is equal to the constant term of $-s(0)^{-1} \langle F, \overline{G_N} \rangle$. The $\text{Aut}(D)$ -invariance of F is necessary in such a formula; see [6] for an alternative construction of \mathbf{G}_N .

That $\Psi(f)$ is a modular form for $\Gamma_0(M)$ follows from the fact that $\Gamma_0(M)/\{\pm I\} \subset \text{O}^+(L)$, where the embedding is given by $\beta \mapsto \alpha\beta\alpha^{-1}$ for $\alpha \in \Gamma_0(M)$.

By Theorem 13.3 of [2], the divisor of $\Psi(f)$ on \mathbb{H} is given by

$$\sum_{\lambda \in L'/\{\pm 1\}, q(\lambda) < 0} s(NQ(\lambda))c(NQ(\lambda))T_{\lambda},$$

where for $\lambda = \begin{pmatrix} a/2M & b/M \\ c & -a/2M \end{pmatrix}$, T_{λ} is the unique solution on \mathbb{H} of the equation

$$Mb\tau^2 - a\tau - c = 0.$$

Therefore, $T_{\lambda} = \tau$ is a CM point and the formula follows. This completes the proof. \square

In practice, the calculation of the Weyl vector ρ is easy, thanks to Corollary 3.3 that determines $H^*(n)$. Alternatively, we may identify $G - G^*$ with a holomorphic modular form by the following proposition, and then determine G^* as well.

Proposition 6.4. We have $G - G^* \in \mathcal{M}(N, \frac{3}{2}, 1)$.

Proof. We noted above that \mathbf{G} and \mathbf{G}^* share the same non-holomorphic part, so $\mathbf{G} - \mathbf{G}^* = G - G^*$ is meromorphic. Since G is holomorphic on \mathbb{H} and at cusps, so is that of G^* . The statement follows. \square

Remark 6.5. We have only considered the case $\chi_p \neq 1$ for each $p \mid M$. To make full use of the isomorphism in Theorem 5.3, we can consider the following lattice

$$L = \left\{ \begin{pmatrix} a & b/M_1 \\ cM_2 & -a \end{pmatrix} : a, b, c \in \mathbb{Z} \right\},$$

with $Q(\alpha) = -M_1 \det(\alpha)$ and $(\alpha, \beta) = M_1 \operatorname{tr}(\alpha\beta)$. Now for $p \mid M_2$, $\chi_p = 1$. Analogous results should hold in this more general setting, but complications may occur at certain places such as the computation of the Weyl vector ρ . The case where $M_1 = 1$ for the Shimura lift is considered in [13]. Even more generally, one may apply our isomorphism to $O(2, n)$ for odd $n > 1$ as well.

7. SOME EXAMPLES

In this section we provide some examples. The simplest case $N = 4$ has been explored extensively by many people. The Zagier duality is worked out by Zagier [21] in this case to give a new proof of Borcherds' theorem. In particular, the Jacobi theta function $\theta = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \in \mathcal{M}^+(4, \frac{1}{2}, 1)$ has Borcherds lift $\eta^2(\tau)$ and that of 12θ is $\Delta(\tau) = \eta^{24}(\tau)$.

Let us first consider the case $N = 12$ and provide some new examples. Using MAGMA, we find the following reduced modular forms for $\mathcal{A}^+(12, \frac{1}{2}, 1)$:

$$\begin{aligned} f_0 &= \frac{1}{2} + q + q^4 + q^9 + O(q^{16}) \\ f_{-3} &= \frac{1}{2}q^{-3} - 7q + 20q^4 - 39q^9 + 84q^{12} - 189q^{13} + O(q^{16}) \\ f_{-8} &= q^{-8} - 34q - 188q^4 + 2430q^9 + 8262q^{12} - 11968q^{13} + O(q^{16}). \end{aligned}$$

For $\mathcal{A}^{\epsilon^*}(12, \frac{3}{2}, 1)$, we have $\epsilon_2^* = -1$, $\epsilon_3^* = -1$, and the basis of reduced modular forms begins with

$$\begin{aligned} f_{-1}^* &= q^{-1} - 1 + 7q^3 + 34q^8 - 22q^{11} - 26q^{12} + O(q^{15}) \\ f_{-4}^* &= q^{-4} - 1 - 20q^3 + 188q^8 + 552q^{11} - 701q^{12} + O(q^{15}) \\ f_{-9}^* &= \frac{1}{2}q^{-9} - 1 + 39q^3 - 2430q^8 + 11178q^{11} - 8826q^{12} + O(q^{15}). \end{aligned}$$

The Zagier duality, i.e. $a_m(-d) = -a_d^*(-m)$, is clear from these two lists.

Since the holomorphic part of Zagier's \mathbf{G} has Fourier expansion

$$G = -\frac{1}{12} + \frac{1}{3}q^3 + \frac{1}{2}q^4 + q^7 + q^8 + q^{11} + O(q^{12}),$$

by Corollary 3.3,

$$G^* = -\frac{1}{6} + \frac{1}{6}q^3 + q^8 + q^{11} + O(q^{12}).$$

Alternatively, the basis of reduced modular forms for $\mathcal{M}(12, \frac{3}{2}, 1)$ consists of

$$\begin{aligned} g_0 &= 1 + 2q^3 + 6q^4 + 12q^7 + O(q^{12}) \\ g_1 &= q + q^3 + 2q^4 + 2q^6 + 2q^7 + q^9 + 4q^{10} + O(q^{12}) \\ g_2 &= q^2 - q^4 + 2q^5 + q^6 - 2q^7 + q^8 + 2q^9 + 2q^{11} + O(q^{12}), \end{aligned}$$

and by Proposition 6.4 and the ϵ^* -condition, we obtain the same expression

$$G^* = G - \frac{1}{12}g_0 = -\frac{1}{6} + \frac{1}{6}q^3 + q^8 + q^{11} + O(q^{12}).$$

For reduced modular forms in $\mathcal{A}^+(12, \frac{1}{2}, 1)$, it is clear that $f_0 = \frac{1}{2}\theta$. It is easy to see that $\rho = \frac{1}{6}$ and since $s(n^2) = 1$ if $3 \nmid n$ and 2 otherwise, we have

$$\Psi(f_0) = q^{\frac{1}{6}} \prod_{n=1}^{\infty} (1 - q^n)^{s(n^2)} = q^{\frac{1}{6}} \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{3n}) = \eta(\tau)\eta(3\tau)$$

is a modular form of weight 1 for $\Gamma_0(3)$ with a character of finite order that is holomorphic and non-vanishing on \mathbb{H} . Similarly,

$$\Psi(f_{-3}) = q^{-\frac{1}{6}} \prod_{n=1}^{\infty} (1 - q^n)^{s(n^2)c(n^2)} = q^{-\frac{1}{6}} (1 - q)^{-7} (1 - q^2)^{20} (1 - q^3)^{-78} (1 - q^4)^{344} \dots$$

is a weakly holomorphic modular form of weight 0 for $\Gamma_0(3)$ which has simple poles at the cusps and simple zeros at the CM-points of discriminant -3 . More explicitly, let

$$E_1 = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + O(q^{12})$$

be the unique modular form of weight 1, level 3 and character $(\frac{\cdot}{3})$ with leading coefficient 1, and then

$$\Psi(f_{-3}) = E_1(\tau)\eta(\tau)^{-1}\eta(3\tau)^{-1}.$$

In other words, $\Psi(f_{-3} + f_0) = E_1$.

We finally add an example for $M = 15$. Now $\epsilon_2^* = -1$, $\epsilon_3^* = -1$ and $\epsilon_5^* = 1$. Similarly, the ϵ^* -component of G (when lifted to level 60) is

$$G^* = -\frac{1}{2} + q^{11} + \frac{1}{2}q^{15} + q^{20} + q^{24} + O(q^{27}).$$

Therefore, the Weyl vector ρ is 1 for $\Psi(f_0)$ and

$$\Psi(f_0) = q \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{3n})(1 - q^{5n})(1 - q^{15n}) = \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau).$$

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