Operator Algebras and Quantum Information

1) Connes implies Tsirelson

2) Optimal states: robust self-testing

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Nonlocal game:

Referee $\Longrightarrow$ Alice and Bob $\Longrightarrow$ Referee:
question sets X, Y $\Longrightarrow$ answer sets A, B

Game rules $=$ winning pairs: (losing pairs $=$ complement)

$$\text{winning} \subseteq (A \times B \mid X \times Y)$$

Strategy $=$ correlation:

$$0 \leq p(ab|xy) \leq 1, \quad \sum_{ab} p(ab|-) \equiv 1.$$

Strategy $=$ winning: $\quad p(\text{losing pairs}) \equiv 0$

Quantum commuting: "Haag-Kastler"

$$p(ab|xy) = \langle\varphi|\, E(a|x)F(b|y)\, |\varphi\rangle$$

Projection valued measures:

$$E(a|x), F(b|y) \in B(H): \quad \sum E(A|-) = 1 = \sum F(B|-)$$

Commuting players:

$$[E(-|-), F(-|-)] = 0$$

Normalized state: $\langle\varphi|\varphi\rangle = 1$

Quantum spatial: "Bohr-Heisenberg"

$$p(ab|xy) = \langle\varphi|\, E(a|x) \otimes F(b|y)\, |\varphi\rangle \quad \Longrightarrow \quad [E \otimes 1, 1 \otimes F] = 0$$

Entangled state: $\varphi \in H \otimes K$

How do Alice and Bob answer: $p(a|x) = ?$ / $q(b|y) = ?$

$$\begin{array}{rlcl} \text{Alice:} & \text{receives } x \in X & \Longrightarrow & \text{answers } a \in A \\ \text{Bob:} & \text{receives } y \in Y & \Longrightarrow & \text{answers } b \in B \end{array}$$

Marginals $=$ non-signaling:

$$p(a|xy) = \sum p(aB|xy) = \langle \varphi | \, E(a|x) \, | \varphi \rangle \equiv p(a|x) \checkmark$$

$$q(b|xy) = \sum p(Ab|xy) = \langle \varphi | \, F(b|y) \, | \varphi \rangle \equiv q(b|y) \checkmark$$

Unified approach: Group algebra

$\implies$ Connes implies Tsirelson

$\implies$ computing quantum values
$\implies$ classifying optimal states

($\implies$ robust self-testing)

Step 1) Commuting families = group algebra:

Single PVM:

$$C^*\left(\sum E(A) = 1\right) = C^*\left(u^A = 1\right) = C^*(\mathbb{Z}/A)$$

Family of PVMs:

$$C^*(\text{player}) := C^*(\mathbb{Z}/A * \ldots * \mathbb{Z}/A)$$

Commuting families:

$$C^*(\text{two player}) := C^*(\text{Alice}) \otimes C^*(\text{Bob})$$

$$A \otimes B = C^*(A \cup B \mid [A, B] = 0)$$

(POVMs $\implies$ operator systems)

Step 2) Quantum strategies = state space:

$$\varphi : C^*(\text{two player}) \to \mathbb{C} : \qquad p(ab|xy) = \varphi\Big(E(a|x) \otimes E(b|y)\Big)$$

Indeed: Stinespring dilation $\Longrightarrow$

$$\begin{tikzcd}
C^*(\text{two player}) \arrow[rr, "\varphi"] \arrow[dr] & & \mathbb{C} \\
& B(\text{Hilb space}) \arrow[ur, "\langle\varphi|-|\varphi\rangle"'] &
\end{tikzcd}$$

Step 3) Later after: Connes $\Longrightarrow$ Tsirelson

Tsirelson's conjecture: Equal correlation classes?

Quantum commuting:

$$C_{qc}(X = Y|A = B) := \left\{p = \varphi(E \otimes E')\right\} \subseteq M(n) \otimes M(m)$$

Quantum spatial:

$$C_{qs}(n|m) \subseteq C_{qc}(n|m) : \qquad C^*(\text{Alice}) \otimes C^*(\text{Bob}) \xrightarrow{-\otimes-} B(H \otimes K) \xrightarrow{\langle\varphi|-|\varphi\rangle} \mathbb{C}$$

Original quantum:

$$C_q(n|m) \subseteq C_{qs}(n|m) : \qquad C^*(\text{Alice}) \otimes C^*(\text{Bob}) \longrightarrow M(d) \longrightarrow \mathbb{C}$$

Quantum approximate:

$$C_{qs}(n|m) \subseteq C_{qa}(n|m) : \qquad C^*(\text{Alice}) \otimes C^*(\text{Bob}) \longrightarrow C^*(\text{Alice}) \otimes_* C^*(\text{Bob}) \longrightarrow \mathbb{C}$$

Tsirelson's conjecture: Equal correlation classes?

Tsirelson's conjecture:

$$C_q(n|m) \subseteq C_{qs}(n|m) \subseteq C_{qa}(n|m) \subseteq C_{qc}(n|m) \subseteq M(n) \otimes M(m):$$

$$\overline{C_q(n|m)} = C_{qa}(n|m) \checkmark \qquad C_{qa}(n|m) = C_{qc}(n|m) \ ?$$

Synchronous correlations:

$$C^s(n|m) := \left\{p(a \neq b|x = y) \equiv 0\right\} \subseteq M(n) \otimes M(m)$$

Synchronous Tsirelson:

$$\text{CEP} \implies \overline{C_q(n|m) \cap C^s(n|m)} = C_{qc}(n|m) \cap C^s(n|m)$$

Proof: elementary $\checkmark$ / MIP*=RE: $\implies \neq \checkmark$

Synchronous games: "Synchronicity rule"

$$A = B, \quad X = Y : \qquad x = y \implies a = b$$

Winning state $\implies$ tracial state:

$$\underline{C^*(\text{two player}) = C^*(\text{player}) \otimes C^*(\text{player}) :}$$

$$\langle\varphi| E(a \neq b|x = y) |\varphi\rangle = 0 \implies E \otimes 1 |\varphi\rangle = 1 \otimes E |\varphi\rangle$$

$$\implies \tau(-) = \varphi(- \otimes 1),\ \varphi(1 \otimes -)$$

Shorthand notation: $\quad E(ab|xy) := E(a|x) \otimes E(b|y)$

$\implies$ group traces: $\quad C^s_{qc}(n|m) = TC^*(n|m) \restriction E(-|-)$!!

Kim–Paulsen–Schafhauser: Elementary Lifting:

$$\begin{tikzcd} & \prod_n M_{d(n)} \arrow[d] \\ C^*\Big(E(1)+\ldots+E(A)=1\Big) \arrow[r] \arrow[ur, dashed] & \mathcal{R}^\omega \end{tikzcd}$$

Tracial ultrapower: Kaplansky: $\mathcal{Q}^\omega = \mathcal{R}^\omega$ (Special thanks to Mikkel !!)

$$\mathcal{Q}^\omega = \prod\nolimits_n \mathcal{Q} \Big/ \Big(\tau_\omega = \lim_{n\to\omega} \tau_n\Big) : \qquad [q]^2 = [q] \implies (q_n \geq 0)$$

$$\mathcal{Q} = M_2 \otimes M_2 \otimes \ldots = \otimes^\infty M_2 : \qquad (q_n \geq 0) \approx (a_n \geq 0) \in M_2 \otimes \ldots \otimes M_2$$

Tracial bound: (optimized)

$$\Big\| a - 1_{(0.5,1]} a \Big\|_2 \leq 2\|a^2 - a\|_2 : \quad (a_n \geq 0) \implies (p_n^2 = p_n = p_n^*) = E(1)$$

Iterations: cutting orthogonal $(E(A) = \text{remainder})$ ✓

CEP $\implies$ synchronous Tsirelson:

$$
\begin{tikzcd}
 & \prod_n M_{d(n)} \arrow[d] \\
C^*(\text{player}) \arrow[ur, dashed, "\text{Lifting}"] \arrow[r, dashed, "\text{CEP}"] \arrow[d] \arrow[dr, phantom, "C_{qc}^s(n|m):\quad \text{some trace}"] & \mathcal{R}^\omega \arrow[d, "\mathrm{tr}_\omega"] \\
\mathbb{C} \arrow[r, equal] & \mathbb{C}
\end{tikzcd}
$$

Our humble contribution: just noticing.

Unified approach: Group algebra

$\implies$ Connes implies Tsirelson ✓

$\implies$ computing quantum values

$\implies$ classifying optimal states

($\implies$ robust self-testing)

Shorthand notation: $E(ab|xy) := E(a|x) \otimes E(b|y)$

Game polynomial:

$$\mathrm{game} := \sum E(\mathrm{winning}) \in C^*(\text{two player}) :$$

$$\varphi(\mathrm{losing}) \equiv 0 \iff \varphi(\mathrm{game}) = |X \times Y|$$

$\Longrightarrow$ quantum value:

$$\omega(\mathrm{game}) = \sup_{\varphi} \varphi(\mathrm{game}) = \|\mathrm{game}\|$$

$\Longrightarrow$ split problem:

$$1) \quad \|\mathrm{game}\| = ?? \quad \Longrightarrow \quad 2) \quad \exists(\varphi = ??) : \quad \varphi(\mathrm{game}) = \|\mathrm{game}\|$$

Fritz–Netzer–Thom: “Can you compute the operator norm?”

MIP*=RE paper: $\Longrightarrow$ Halting problem !!!

Minimal example: CHSH game:

$$A = \{0,1\} = B, \quad X = \{0,1\} = Y :$$

$$\text{winning} : \; a \oplus b = x \cdot y$$

Two-player algebra:

$$C^* \langle u, v | u^2 = 1 = v^2 \rangle \otimes C^* \langle u, v | u^2 = 1 = v^2 \rangle$$

Bias polynomial:

$$\mathrm{CHSH} := u \otimes u + u \otimes v + v \otimes u - v \otimes v$$

Operator norm: Landau's trick!

Step 1) algebraic: (Fritz–Netzer–Thom + random walks)

$$\mathrm{CHSH}^*\mathrm{CHSH} = 4 - [u,v] \otimes [u,v] \leq 8$$

Step 2) good repr: $C^*(\text{two player}) \to B(H)$: $\|\mathrm{CHSH}^*\mathrm{CHSH}\| = 8$

$$\text{Quantum value} = \frac{\|\mathrm{game}\|}{|X \times Y|} = \frac{1}{2}\left(1 + \frac{\|\mathrm{bias}\|}{|X \times Y|}\right) = \frac{1}{2} + \frac{1}{2\sqrt{2}} \geq 75\%$$

[See preprint]: optimal state = single solution: ($>$ self-testing!)

$\implies$ anticommutation: $\varphi(\mathrm{CHSH}) = 2\sqrt{2} \implies$

$$\{u,v\} \otimes 1 \,|\varphi\rangle = 0 = 1 \otimes \{u,v\}\,|\varphi\rangle \underset{\{u,v\} \in \mathcal{Z}C^*(u^2=1=v^2)}{\implies} \{u,v\} \otimes 1 = 0 = 1 \otimes \{u,v\}$$

$\implies$ factorization: $\mathrm{M}_2 = \frac{C^*(u,v)}{\{u,v\}=0}$ $\implies$ single solution: $\varphi : C^*(\ldots) \to \mathbb{C}$:

$$\begin{tikzcd}
C^*(\text{two player}) \arrow[rr] \arrow[dr] & & \mathbb{C} \\
& \mathrm{M}_2 \otimes \mathrm{M}_2 \arrow[ur] &
\end{tikzcd}$$

$\varphi(\ldots)$	$-\otimes 1$	$-\otimes u$	$-\otimes v$	$-\otimes uv$
$1 \otimes -$	1	0	0	0
$u \otimes -$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
$v \otimes -$	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0
$uv \otimes -$	0	0	0	-1

Compare $\implies$ Correlation table: $p(ab, xy) =$

$$\varphi(u\otimes 1), \quad \varphi(v \otimes 1), \quad \varphi(1 \otimes u), \quad \varphi(1 \otimes v),$$
$$\varphi(u \otimes u), \quad \varphi(u \otimes v), \quad \varphi(v \otimes u), \quad \varphi(v \otimes v), \quad \cancel{\varphi(uv \otimes vuv)}, \quad \text{etc}$$

[LLP10] and [AMP12]: Tilted CHSH games:

$$\mathrm{CHSH}(\alpha, \beta \in \mathbb{R}) = \alpha u \otimes (u + v) + v \otimes (u - v) + 2\beta u \otimes 1$$

$\Longrightarrow$ basic building blocks:

I3322 inequality $\Longrightarrow$ fin dim / spatial ?

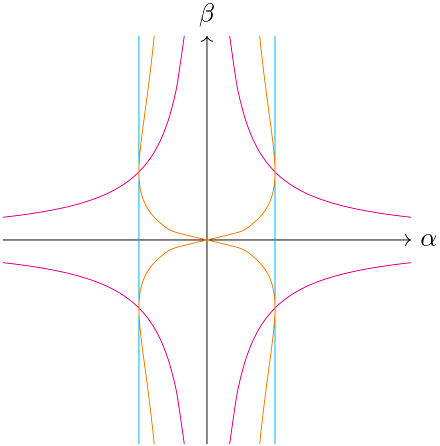
Pal–Vertesi, Coladangelo–Stark, etc

Squared bias = tedious:

$$\mathrm{CHSH}(\alpha, \beta)^2 = \dots \Longrightarrow \text{?!?}$$

$\Longrightarrow$ need systematic approach!

Teaser from [F–Shahiri]: repr theory ! $\implies$ quantum regions:



Unravel non-symmetry: $\|\mathrm{CHSH}(\alpha, -)\| \neq \|\mathrm{CHSH}(1/\alpha, -)\|$ (cf. [AMP12])

Optimal state $\Longrightarrow$ minimal quotient: $\varphi\Big(\mathrm{CHSH}(\alpha,\beta)\Big)=\|\mathrm{CHSH}(\alpha,\beta)\| \quad \Longrightarrow$

$$C^*(\text{Alice})\otimes C^*(\text{Bob}) \longrightarrow C^*\Big(\{u,v\}=2s\Big)\otimes C^*\Big(\{u,v\}=2t\Big) \longrightarrow \mathbb{C}$$

Theorem (F–Shahiri):

Solution uniquely and entirely classified (including all higher moments) by

$\varphi(\ldots)$	$-\otimes 1$	$-\otimes u$	$-\otimes v$	$-\otimes uv$
$1\otimes -$	$1=a^2+d^2$	$(a^2-d^2)\lvert W_+\rvert$	$(a^2-d^2)\lvert W_+\rvert$	$(a^2-d^2)(\lvert W_+\rvert^2-\lvert W_-\rvert^2)$
$u\otimes -$	a^2-d^2	$(a^2+d^2)\lvert W_+\rvert$	$(a^2+d^2)\lvert W_+\rvert$	$(a^2+d^2)(\lvert W_+\rvert^2-\lvert W_-\rvert^2)$
$v\otimes -$	0	$2ad\lvert W_-\rvert$	$-2ad\lvert W_-\rvert$	0
$uv\otimes -$	0	0	0	$2ad\lvert W_+\rvert\cdot\lvert W_-\rvert$

where (to be continued on next slide ...)

Theorem (F–Shahiri) continued:

where for shorthand $1 \otimes W_{\pm} := 1 \otimes (u \pm v)$: $\quad |W_+|=2|\alpha|\sqrt{\frac{1+\beta^2}{1+\alpha^2}}, \quad |W_-|=2\sqrt{\frac{1-\alpha^2\beta^2}{1+\alpha^2}}$

and coefficients given by

$$a=\frac{1}{\sqrt{2}}\sqrt{\frac{1-\alpha^2\beta^2}{\sqrt{1+\beta^2}-\beta\sqrt{1+\alpha^2}}}, \quad d=\frac{1}{\sqrt{2}}\sqrt{\frac{1}{\sqrt{1+\beta^2}}}$$

from which there is no ambiguity left anymore!
The optimal state is necessarily a vector state and given by the 1-dim projections

$$\varphi(-)=\langle\varphi|-|\varphi\rangle: \quad |\varphi\rangle\langle\varphi|=aP(u=1)\otimes P\left(\frac{W_+}{|W_+|}=1\right)+dP(u=-1)\otimes P\left(\frac{W_+}{|W_+|}=-1\right)$$

with Schmidt coefficients from above and so always partially entangled (nonlocal),

$$\mathrm{rk}(|\varphi\rangle) = 2 \iff a \text{ and } d \neq 0,$$

and maximally entangled only along the horizontal axis:

$$a = d \iff (\alpha \neq 0, \beta = 0).$$

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Thank you for your attention.

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