The Parabolic Flows for Complex Quotient Equations

Wei Sun

ShanghaiTech University

Introduction

(M^n, ω) – a compact Kähler manifold of complex dimension n
without boundary (closed);

χ – a smooth closed real (1, 1) form in Γ^k_ω.

Γ^k_ω is the set of all the real $(1, 1)$ forms whose eigenvalue sets with respect to ω belong to k-positive cone in $\mathbb{R}^n$.

Introduction

We study the parabolic equations

$$\frac{\partial u}{\partial t} = \log \frac{\chi_u^k \wedge \omega^{n-k}}{\chi_u^l \wedge \omega^{n-l}} - \log \psi, \tag{1}$$

where $\psi \in C^\infty(M)$, $0 \leq l < k \leq n$ and

$$\chi_u := \chi + \frac{\sqrt{-1}}{2}\partial\bar{\partial}u. \tag{2}$$

To be nondegenerate elliptic, we seek the admissible solution u such that $\chi_u \in \Gamma_\omega^k$. Thus, we need to assume $\Psi > 0$.

Introduction

The study of the parabolic flows is motivated by complex equations

$$\chi_u^k \wedge \omega^{n-k} = \psi \chi_u^l \wedge \omega^{n-l}, \qquad \chi_u \in \Gamma_\omega^k. \tag{3}$$

When ψ is constant, it must be c defined by

$$c := \frac{\int_M \chi^k \wedge \omega^{n-k}}{\int_M \chi^l \wedge \omega^{n-l}}. \tag{4}$$

This is an extension of complex Monge-Ampère equation [Cao, 1985] and complex Monge-Ampère type equation [Sun, 2015].

- In 1978, Yau proved the Calabi conjecture by solving the complex Monge-Ampère equations on closed Kähler manifolds.

- In 1978, Yau proved the Calabi conjecture by solving the complex Monge-Ampère equations on closed Kähler manifolds.
- In 1985, Cao reproduced the result by parabolic flow method.

- In 1978, Yau proved the Calabi conjecture by solving the complex Monge-Ampère equations on closed Kähler manifolds.
- In 1985, Cao reproduced the result by parabolic flow method.
- In 1987 and 2010, Cherrier, Tosatti and Weinkove independently solved the equation on closed Hermitian manifolds under the balanced condition.

- In 1978, Yau proved the Calabi conjecture by solving the complex Monge-Ampère equations on closed Kähler manifolds.
- In 1985, Cao reproduced the result by parabolic flow method.
- In 1987 and 2010, Cherrier, Tosatti and Weinkove independently solved the equation on closed Hermitian manifolds under the balanced condition.
- Later in 2010, Tosatti and Weinkove successfully removed the balanced condition and extended the result to general Hermitian manifolds.

- In 1978, Yau proved the Calabi conjecture by solving the complex Monge-Ampère equations on closed Kähler manifolds.
- In 1985, Cao reproduced the result by parabolic flow method.
- In 1987 and 2010, Cherrier, Tosatti and Weinkove independently solved the equation on closed Hermitian manifolds under the balanced condition.
- Later in 2010, Tosatti and Weinkove successfully removed the balanced condition and extended the result to general Hermitian manifolds.
- In 2011, Gill gave a parabolic proof for the result.

When χ, ω are both Kähler and ψ is a constant:

$$\psi = \frac{\int_M \chi^n}{\int_M \chi^{n-k} \wedge \omega^k}.$$

- In 2004, Chen used the parabolic flow to study the equation, i.e. J-flow.

When χ, ω are both Kähler and ψ is a constant:

$$\psi = \frac{\int_M \chi^n}{\int_M \chi^{n-k} \wedge \omega^k}.$$

- In 2004, Chen used the parabolic flow to study the equation, i.e. J-flow.
- In 2006, Weinkove solved the equation by J-flow.

When χ, ω are both Kähler and ψ is a constant:

$$\psi = \frac{\int_M \chi^n}{\int_M \chi^{n-k} \wedge \omega^k}.$$

- In 2004, Chen used the parabolic flow to study the equation, i.e. J-flow.
- In 2006, Weinkove solved the equation by J-flow.
- In 2008, Song and Weinkove proved a necessary and sufficient condition for solvability, i.e. the cone condition.

When χ, ω are both Kähler and ψ is a constant:

$$\psi = \frac{\int_M \chi^n}{\int_M \chi^{n-k} \wedge \omega^k}.$$

- In 2004, Chen used the parabolic flow to study the equation, i.e. J-flow.
- In 2006, Weinkove solved the equation by J-flow.
- In 2008, Song and Weinkove proved a necessary and sufficient condition for solvability, i.e. the cone condition.
- In 2011, Fang, Lai and Ma extended the cone condition and the solvability to all $1 \leq k < n$.

Condition

Subsolution condition is too strong for closed manifolds!

We study a priori estimates and the convergence under the cone condition, that is,

there is a real-valued C^2 function $\underline{u}$ satisfying $\chi_{\underline{u}} \in \Gamma_\omega^k$ and

$$k\chi_{\underline{u}}^{k-1} \wedge \omega^{n-k} > l\psi\chi_{\underline{u}}^{l-1} \wedge \omega^{n-l}.$$

For convenience, we adopt an equivalent definition of $\underline{u}$ due to Székelyhidi, which is called $\mathcal{C}$-subsolution.

We say that a C^2 function $\underline{u}$ is a $\mathcal{C}$-subsolution if $\chi_{\underline{u}} \in \Gamma^k_\omega$, and at each point $x \in M$, the set

$$\left\{ \tilde{\chi} \in \Gamma^k_\omega \,\middle|\, \tilde{\chi}^k \wedge \omega^{n-k} \leq \psi \tilde{\chi}^l \wedge \omega^{n-l} \text{ and } \tilde{\chi} - \chi_{\underline{u}} \geq 0 \right\} \quad (5)$$

is bounded.

Main Result

Let (M^n, ω) be a closed Kähler manifold of complex dimension n and χ a smooth closed real $(1,1)$ form in Γ^k_ω. Suppose that there is a $\mathcal{C}$-subsolution $\underline{u}$ and $\psi \geq c$ for all $x \in M$. Then there exists a long time solution u. Moreover, the normalization $\hat{u}$ of u is C^∞ convergent to a smooth function $\hat{u}_\infty$ where $\hat{u}$ is defined later. Consequently, there is a unique real number b such that the pair $(\hat{u}_\infty, b)$ solves

$$\frac{\chi_u^k \wedge \omega^{n-k}}{\chi_u^l \wedge \omega^{n-l}} = e^b \psi. \tag{6}$$

J-functional

It is easy to see that for general ψ, the solution u is probably divergent. It is necessary to find an appropriate normalization. We adapt the general J-functionals [Chen, 2000; Fang-Lai-Ma, 2011]. Let $\mathcal{H}$ be the space

$$\mathcal{H} := \{u \in C^\infty(M) \mid \chi_u \in \Gamma^k_\omega\}. \tag{7}$$

For any curve $v(s) \in \mathcal{H}$, we define the funtional J_l by

$$J_l(u) = \int_0^1 \int_M \frac{\partial v}{\partial s} \chi_v^l \wedge \omega^{n-l} ds, \tag{8}$$

where $v(s)$ is an arbitrary path in $\mathcal{H}$ connecting 0 and u.

Those functionals are independent from choices of the path.

Along the solution flow $u(x,t)$, we have

$$\begin{aligned}\frac{d}{dt}J_l(u) &= \int_M \left(\log\frac{\chi_u^k\wedge\omega^{n-k}}{\chi_u^l\wedge\omega^{n-l}} - \log\psi\right)\chi_u^l\wedge\omega^{n-l}\\ &\leq \log c\int_M \chi_u^l\wedge\omega^{n-l} - \int_M \log\psi\chi_u^l\wedge\omega^{n-l}\\ &\leq 0.\end{aligned} \tag{9}$$

Let

$$\hat{u} = u - \frac{J_l(u)}{\int_M \chi^l\wedge\omega^{n-l}}. \tag{10}$$

By (9), we know that $\partial_t\hat{u} \geq \partial_t u$.

L^∞ estimate

We claim that

$$\inf_M (\hat{u} - \underline{u})(x,t) > -2 \sup_{M\times\{0\}} |\partial_t u| - C_0, \tag{11}$$

where $C_0 \geq 0$ is to be determined later. Otherwise, there must be time $t_0 > 1$ such that

$$\inf_M (\hat{u} - \underline{u})(x,t_0) = \inf_{M\times[0,t_0]} (\hat{u} - \underline{u})(x,t) = -2 \sup_{M\times\{0\}} |\partial_t u| - C_0. \tag{12}$$

Let $v = \hat{u} - \underline{u} - \epsilon + \epsilon|z|^2 - \epsilon(t - t_0) - \inf_M(\hat{u} - \underline{u})(x, t_0)$ for some small $\epsilon > 0$.
We may assume that $\epsilon < \lambda$. It is easy to see that when $t = t_0 - 1$

$$v = \hat{u} - \underline{u} + \epsilon|z|^2 - \inf_M(\hat{u} - \underline{u})(x, t_0) \geq 0, \tag{13}$$

and when $|z|^2 = 1,\ t \leq t_0$

$$v = \hat{u} - \underline{u} - \epsilon(t - t_0) - \inf_M(\hat{u} - \underline{u})(x, t_0) \geq 0. \tag{14}$$

Moreover,

$$\inf_{M\times[t_0-1,t_0]} v = \inf_{M\times\{t_0\}} v = v(x_0, t_0) = -\epsilon. \tag{15}$$

ϵ is chosen small enough, we obtain an bound $|u_{i\bar{j}}| < C$ in Γ_{-v}.

L^∞ estimate

By Alexandroff-Bakelman-Pucci maximum principle, we have

$$\begin{aligned}\epsilon &\leq C\left[\int_{\Gamma_{-v}\cap\{v<0\}} -\partial_t v \det(D_x^2 v)\,dxdt\right]^{\frac{1}{2n+1}} \\ &\leq C\left[\int_{\Gamma_{-v}\cap\{v<0\}} -\partial_t v 2^{2n}(\det(v_{i\bar{j}}))^2\,dxdt\right]^{\frac{1}{2n+1}}.\end{aligned} \tag{16}$$

Because of the boundedness of $u_{i\bar{j}}$ and $\partial_t u$, it follows that

$$\epsilon \leq C\left|\Gamma_{-v}\cap\{v<0\}\right|^{\frac{1}{2n+1}}. \tag{17}$$

L^∞ estimate

So

$$\begin{aligned}
\epsilon^{2n+1} &\leq C\left|M\times[t_0-1,t_0]\cap\left\{\hat{u}<\inf_M(\hat{u}-\underline{u})(x,t_0)\right\}\right| \\
&\leq C\int_{t_0-1}^{t_0}\frac{||\hat{u}^-(x,t)||_{L^1}}{|\inf_M(\hat{u}-\underline{u})(x,t_0)|}dt \\
&\leq C\int_{t_0-1}^{t_0}\frac{||\hat{u}(x,t)-\sup_M\hat{u}(x,t)||_{L^1}}{|\inf_M(\hat{u}-\underline{u})(x,t_0)|}dt
\end{aligned} \tag{18}$$

C^2 estimate

There exists a constant C depending on $\sup_{M\times[0,T)}|\hat{u}|$ such that for any $t' \in [0, T)$,

$$\sup_M |\partial\bar{\partial}u| \le C\left(\sup_{M\times[0,t']} |\nabla u|^2 + 1\right), \tag{19}$$

at any time $t \in [0, t']$.

C^2 estimate

Following the work of [Hou-Ma-Wu, 2010], we define

$$H(x,\xi) = \log\Big(\sum_{i,j} X_{i\bar{j}}\xi^i\bar{\xi}^j\Big) + \varphi(|\nabla u|^2) + \rho(\hat{u} - \underline{u}) \qquad (20)$$

where

$$\begin{aligned}
\varphi(s) &= -\frac{1}{2}\log\Big(1 - \frac{s}{2K}\Big), \qquad \text{for } 0 \le s \le K-1, \\
\rho(t) &= -A\log\Big(1 + \frac{t}{2L}\Big), \qquad \text{for } -L+1 \le t \le L-1,
\end{aligned} \qquad (21)$$

with

$$\begin{aligned}
K &:= \sup_{M\times[0,t']} |\nabla u|^2 + \sup |\nabla \underline{u}|^2 + 1, \\
L &:= \sup_{M\times[0,T)} |\hat{u}| + \sup_{M} |\underline{u}| + 1, \\
A &:= 3L(C_0 + 1)
\end{aligned}$$

and C_0 is to be specified.

Lemma

There is a constant $\theta > 0$ such that we have either

$$\sum_i F^{i\bar{i}}(u_{i\bar{i}} - \underline{u}_{i\bar{i}}) \leq F(\chi_u) - \log\Psi - \theta\Big(1 + \sum_i F^{i\bar{i}}\Big), \tag{22}$$

or

$$F^{j\bar{j}} \geq \theta\Big(1 + \sum_i F^{i\bar{i}}\Big), \qquad \forall j = 1, \cdots, n. \tag{23}$$

C^2 estimate

Without loss of generality, we may assume that $X_{1\bar{1}} \geq \cdots \geq X_{n\bar{n}}$. Thus

$$F^{n\bar{n}} \geq \cdots \geq F^{1\bar{1}}. \tag{24}$$

If $\lambda > 0$ is small enough, $\chi - \lambda\omega$ and $\underline{u}$ still satisfy the definition of $\mathcal{C}$-subsolution.
Since M is compact, there are uniform constants $N > 0$ and $\sigma > 0$ such that

$$F(\chi') > \log \Psi + \sigma, \tag{25}$$

where

$$\chi' = \chi_{\underline{u}} - \lambda g + \left\{ \begin{matrix} N & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{matrix} \right\}_{n \times n}. \tag{26}$$

Direct calculation shows that

$$\begin{aligned}\sum_i F^{i\bar{i}}(u_{i\bar{i}} - \underline{u}_{i\bar{i}}) &= \sum_i F^{i\bar{i}} X_{i\bar{i}} - \sum_i F^{i\bar{i}} \chi'_{i\bar{i}} + NF^{1\bar{1}} - \lambda \sum_i F^{i\bar{i}} \\ &\leq F(\chi_u) - F(\chi') + NF^{1\bar{1}} - \lambda \sum_i F^{i\bar{i}} \\ &\leq F(\chi_u) - \log \Psi - \sigma - \lambda \sum_i F^{i\bar{i}} + NF^{1\bar{1}}.\end{aligned} \tag{27}$$

If

$$\frac{\min\{\sigma, \lambda\}}{2}\Big(1 + \sum_i F^{i\bar{i}}\Big) \geq NF^{1\bar{1}}, \tag{28}$$

we obtain (22); otherwise, inequality (23) has to be true.

Moreover, when (22) holds true, we have

$$\begin{aligned}\sum_i F^{i\bar{i}}(u_{i\bar{i}} - \underline{u}_{i\bar{i}}) &\leq F(\chi_u) - \log\Psi - \theta\Big(1 + \sum_i F^{i\bar{i}}\Big) \\ &= \partial_t u - \theta\Big(1 + \sum_i F^{i\bar{i}}\Big) \\ &\leq \partial_t \hat{u} - \theta\Big(1 + \sum_i F^{i\bar{i}}\Big).\end{aligned} \tag{29}$$

When (23) holds true, we use the fact that

$$\sup|\partial_t \hat{u}| \leq 2 \sup_M |\partial_t u(x,0)| \tag{30}$$

Thanks !