Self-similar solutions and umbilic hypersurfaces

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Mean curvature flow and Gauss curvature flow

- $X_0 : M^n \to \mathbb{R}^{n+1}$ a smooth hypersurface in $\mathbb{R}^{n+1}$
- F-Curvature Flow: a family of smooth immersions
  $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ satisfying

$$\begin{cases} \frac{\partial}{\partial t} X(x, t) = -F(x, t)\nu(x, t), \\ X(x, 0) = X_0(x), \end{cases}$$

  where F is a smooth function of the principal curvatures of $M_t = X(M, t)$ and ν is the outward unit normal of M_t.

- If $F = H$, then X is called the *mean curvature flow*.
- If $F = K$, then X is called the *Gauss curvature flow*.

Theorem (Huisken, 1984)

If X_0 is a closed strictly convex hypersurface, then the MCF with X_0 has a smooth solution $X(\cdot, t)$ on a maximal finite time interval $[0, T)$ and $X(\cdot, t)$ converges to a single point as $t \to T$. Moreover, the hypersurface $X(\cdot, t)$ converges to a round sphere after appropriate rescaling.

- Tso (1985): Any closed convex hypersurfaces evolving by the Gauss curvature flow converges to a single point.
- Chow (1985): same for $F = K^\alpha$ $(\alpha > 0)$; Moreover, for $\alpha = \frac{1}{n}$, $X(\cdot, t)$ converges to a round sphere after appropriate rescaling.

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σ_k-curvature

- $\kappa = (\kappa_1, \kappa_2, ..., \kappa_n)$ principal curvature of M
- σ_k: the k-th elementary symmetric functions of κ, i.e.,

$$\sigma_k(\kappa) = \sum_{1 \le i_1 < i_2 \cdots < i_k \le n} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_k}.$$

- $H = \sigma_1(\kappa),\ K = \sigma_n(\kappa)$.

σ_k^α-curvature flows

- Under certain convexity condition or pinching condition of the principal curvatures, a closed hypersurface evolving by the following F-curvature flows converges to a round point.
  - $F = \sigma_2^{\frac{1}{2}}$ (Chow 1987)
  - $F = \sigma_k^{\frac{1}{k}}$ (Andrews 1994, 2007)
  - $F = \sigma_1^\alpha$ (Schulze 2005, 2006)
  - $F = \sigma_2^\alpha$ (Alessandroni-Sinestrari 2010), ...

We call $F = \sigma_k^\alpha$ curvature flow σ_k^α-curvature flow.

Self-similar solutions of the σ_k^α-curvature flow

Let $X : M \to \mathbb{R}^{n+1}$ be a smooth embedding of a closed, orientable hypersurface satisfying

$$(**) \qquad \sigma_k^\alpha = \langle X, \nu \rangle.$$

Then it is called a *self-similar solution* of the σ_k^α-curvature flow.

- If X is a solution of (**), then

$$\tilde{X}(x,t) = ((k\alpha+1)(T-t))^{\frac{1}{1+k\alpha}} X(x)$$

  becomes a solution of the σ_k^α-curvature flow up to a tangential diffeomorphism. So we call the solutions of (**) self-similar solutions of the σ_k^α-curvature flow.

Type-I Finite Time Singularities in MCF and self-similar solutions

- $p \in \mathbb{R}^{n+1}$ is called a *singularity* of MCF as $t \to T < \infty$, if $\exists x \in M$ s.t. $X(x,t) \to p$ and $|A|^2(x,t) \to \infty$ as $t \to T$.
- We define a *Finite Time Singularity* in MCF to be
  - type I singularity: $\sup_{M \times [0,T)} |A(\cdot,t)|(T-t)^{1/2} < \infty$,
  - type II singularity: otherwise.

Huisken's monotonicity formulae implies

Theorem (Huisken 1990)

The solutions of MCF forming a type-I singularity can be homothetically rescaled such that any resulting limiting hypersurface is a self-similar solution.

Variational characterization of self-similar solutions of MCF

Define

$$\mathscr{F}_{x_0,t_0}(M) := (4\pi t_0)^{-\frac{n}{2}} \int_M e^{-\frac{|x-x_0|^2}{4t_0}}\, d\mu.$$

Theorem (Colding-Minicozzi, 2012)

M is a critical point of $\mathscr{F}_{x_0,t_0}$ iff

$$H = \frac{\langle X - x_0, \nu \rangle}{2t_0}.$$

Closed self-similar solutions of MCF

Theorem (Huisken,1990)

If M is a closed hypersurface in $\mathbb{R}^{n+1}$, with $H \geq 0$ and satisfies

$$H = \langle X, \nu \rangle,$$

then M must be a round sphere.

Asymptotic behavior of α-Gauss curvature flow

For $F = K^\alpha$, we call the F-curvature flow α-Gauss curvature flow.

When $\alpha = 1$ and $n = 2$,

(Firey conjecture, proved by Andrews 1999)

Any closed strictly convex surface in $\mathbb{R}^3$ evolving the Gauss curvature flow converges to a round point.

Andrews (1996): When $\alpha = \frac{1}{n+2}$, the rescaled solution converges to an ellipsoid.

Asymptotic to a self-similar solution

The α-Gauss curvature flow converges to a self-similar solution after normalization in the following cases

- $\frac{1}{n+2} < \alpha < \frac{1}{n}$, Andrews (2000)
- $\alpha = 1$, Guan-Ni (2013)
- $\alpha > \frac{1}{n+2}$, Andrews-Guan-Ni (2016)

- Then the characterization of asymptotic behavior of α-Gauss curvature flow is reduced to the study on self-similar solutions.

Self-similar solutions of α-Gauss curvature flow

- Choi-Daskalopoulos (2016): when $\alpha \in (\frac{1}{n}, 1 + \frac{1}{n})$, the closed strictly convex self-similar solution of the α-Gauss curvature flow is a round sphere.

Theorem (Brendle-Choi-Daskalopoulos, 2017)

Let M be a closed, strictly convex hypersurface in $\mathbb{R}^{n+1}$ satisfying

$$K^\alpha = \langle X, \nu \rangle.$$

If $\alpha > \frac{1}{n+2}$, then M must be a round sphere; If $\alpha = \frac{1}{n+2}$, then M is an ellipsoid.

- Combining with Andrews-Guan-Ni's result, this gives asymptotic behavior of α-Gauss curvature flow.

Question

From these theorems, the following natural question arises:

Question

Let M be a closed, strictly convex hypersurface in $\mathbb{R}^{n+1}$ satisfying

$$\sigma_k^\alpha = \langle X, \nu \rangle,$$

where $1 \leq k \leq n-1$, $\alpha \geq \frac{1}{k}$. Can we conclude that M must be a round sphere?

Affirmative answer-Main Results I

Theorem 2.1 (Gao-Li-M. 2018).

Let M be a closed, strictly convex hypersurface in $\mathbb{R}^{n+1}$ satisfying

$$\sigma_k^\alpha = \langle X, \nu \rangle,$$

where $1 \leq k \leq n-1$, $\alpha \geq \frac{1}{k}$. Then M must be a round sphere.

In fact, we prove similar results for more general speed function F.

More general F

Condition $(*)$

Suppose F is a symmetric function defined on the positive cone $\Gamma_+ = \{\kappa \in \mathbb{R}^n | \kappa_1 > 0, \kappa_2 > 0, \cdots, \kappa_n > 0\}$, and

i) $F > 0$ and $\frac{\partial F}{\partial \lambda_i} > 0$ for $1 \leq i \leq n$.

ii) $F(t\lambda) = t^\beta F(\lambda)$, $\forall t \in \mathbb{R}_+$.

iii) $\forall i \neq j$,

$$\frac{\frac{\partial F}{\partial \kappa_i}\kappa_i - \frac{\partial F}{\partial \kappa_j}\kappa_j}{\kappa_i - \kappa_j} \geq 0.$$

iv) $\forall (y_1, ..., y_n) \in \mathbb{R}^n$,

$$\sum_i \frac{1}{\kappa_i} \frac{\partial \log F}{\partial \kappa_i} y_i^2 + \sum_{i,j} \frac{\partial^2 \log F}{\partial \kappa_i \partial \kappa_j} y_i y_j \geq 0.$$

Remark on Condition $(*)$

- Both H^α and K^α satisfy i) – iii) in Condition $(*)$.
  - Cauchy-Schwarz inequality $\Rightarrow$ iv) holds for $F = H^\alpha$.
  - iv) holds for $F = K^\alpha$ trivially.
- Easy to check i) – iii) for σ_k^α and iv) is equivalent to

$$\sum_{i=1}^n \frac{\sigma_{k-1;i}}{\kappa_i \sigma_k} y_i^2 + \sum_{i \neq j} \frac{\sigma_{k-2;ij}}{\sigma_k} y_i y_j \geq \Big(\sum_{i=1}^n \frac{\sigma_{k-1;i}}{\sigma_k} y_i\Big)^2,$$

  which also plays an important role in Guan-Ma's work (2003) on the Christoffel-Minkowski problem.
- Condition $(*)$ also holds for $S_k(\kappa) = \sum_{i=1}^n \kappa_i^k$.
- In fact, any multiplication combination of such functions, such as $\sigma_2 \sigma_3$ satisfies Condition $(*)$.
- iii) and iv) are equivalent to the convexity of the function $F^*(A) = \log F(e^A)$ defined on real $n \times n$ symmetric matrices.
- Andrews studied convex and inverse convex functions, i.e., both F and $F^*(A) = -F(A^{-1})$ convex, which do not include σ_k.

Main result II

For such general F, we prove

Theorem 2.2 (Gao-Li-M. 2018).

Let M be a closed, strictly convex hypersurface in $\mathbb{R}^{n+1}$ satisfying

$$\langle X, \nu \rangle = F - C,$$

with constant C. For $\beta > 1$ and $C \geq 0$, if F satisfies Condition $(*)$, then M must be a round sphere.

Self-similar solutions in a warped product space

- Self-similar solutions of the MCF were introduced in warped product manifolds (Futaki-Hattori-Yamamoto 2014, Wu 2017, Alias-Lira-Rigoli 2017).

- Let $N = [0, \bar{r}) \times \mathbb{S}^n$ be a warped product manifold with metric
$$\bar{g} = dr^2 + \lambda^2(r) g_{\mathbb{S}},$$
where λ is a non-negetive function of r and $g_{\mathbb{S}}$ is the standard metric of $\mathbb{S}^n$.

- If a hypersurface $X : M^n \to N^{n+1}$ in a warped product N satisfies
$$\bar{g}\left(\lambda(r(x))\partial_r(x), \nu(x)\right) = F(\kappa(x)),$$
then it is called a self-similar solution of the curvature flow
$$\frac{\partial}{\partial t}\tilde{X} = -F(\kappa)\nu.$$

For a warped product N, when the warping factor $\lambda(r) = r$, $\sin r$, or $\sinh r$, N is the Euclidean space $\mathbb{R}^{n+1}$, the sphere $\mathbb{S}^{n+1}$ or the hyperbolic space $\mathbb{H}^{n+1}$ with constant sectional curvature $\epsilon = 0$, 1 or -1 respectively.

Theorem 2.3 (Gao-M.,2019).

Let M be a closed, strictly convex hypersurface in the hemisphere $\mathbb{S}^{n+1}_+$ satisfying

$$\bar{g}(\lambda\partial_r, \nu) = F - C.$$

For $\beta \geq 1$ and $C \geq 0$, if F satisfies Condition $(*)$, then M is a slice $\{r_0\} \times \mathbb{S}^n$ in $\mathbb{S}^{n+1}_+$.

Self-similar solutions in the hemisphere

In $\mathbb{S}^{n+1}_+$, the self-similar solution of α-Gauss curvature flow is umbilic even for $\alpha = \frac{1}{n+2}$.

Corollary 2.4 (Gao-M., 2019).

Let M be a closed, strictly convex hypersurface in the hemisphere $\mathbb{S}^{n+1}_+$ satisfying

$$\bar{g}(\lambda \partial_r, \nu) = \sigma_n^\alpha(\kappa) - C.$$

If $\alpha \geq \frac{1}{n+2}$ and $C \geq 0$, then M is a slice $\{r_0\} \times \mathbb{S}^n$ in $\mathbb{S}^{n+1}_+$.

Self-similar solutions in hyperbolic space

For self-similar solutions to a relevant curvature flow in $\mathbb{H}^3$, we obtain

Theorem 2.5 (Gao-M., 2019).

Let M be a closed, strictly convex surface in $\mathbb{H}^3$ satisfying

$$\bar{g}(\lambda\partial_r, \nu) = K - C. \tag{2.1}$$

If $C \geq 1$, then M is a slice $\{r_0\} \times \mathbb{S}^2$ in $\mathbb{H}^3$.

Fundamental formulas of self-similar solutions

- Define the operator $\mathcal{L}$ by

$$\mathcal{L} = \frac{\partial F}{\partial h_{ij}} \nabla_i \nabla_j.$$

- Denote $\Phi = \int_0^r \lambda(s)\, ds$.

  From the equation

$$\bar{g}(\lambda \partial_r, \nu) = F - C,$$

we get

$$\mathcal{L}F = \bar{g}(\lambda \partial_r, \nabla F) + \beta \lambda' F - \frac{\partial F}{\partial h_{ij}} h_{il} h_{jl} (F - C) + \frac{\partial F}{\partial h_{ij}} \bar{R}_{\nu j l i} \bar{g}(\lambda \partial_r, e_l),$$

$$\mathcal{L}\Phi = \lambda' \sum_i \frac{\partial F}{\partial h_{ii}} - \beta F(F - C).$$

Test function

Let (b^{ij}) denote the inverse of (h_{ij}). Set

$$Z = F\,\mathrm{tr}\,b - \frac{n(\beta-1)}{\beta}\Phi.$$

$$\begin{aligned}
\mathcal{L}Z = {} & 2\frac{\partial F}{\partial h_{ij}}\nabla_i F \nabla_j \mathrm{tr}\,b + \bar{g}(\lambda\partial_r, \nabla(F\mathrm{tr}\,b)) + (\beta-1)\lambda'(F\mathrm{tr}\,b - \frac{n}{\beta}\sum_i \frac{\partial F}{\partial h_{ii}}) \\
& + C(\frac{\partial F}{\partial h_{ij}} h_{il}h_{jl}\mathrm{tr}\,b - \beta n F) + F b^{kp} b^{qk} \frac{\partial^2 F}{\partial h_{ij}\partial h_{st}} h_{ijp}h_{stq} \\
& + 2F b^{ks} b^{pt} b^{kq} \frac{\partial F}{\partial h_{ij}} h_{sti}h_{pqj} + (\mathrm{tr}\,b\frac{\partial F}{\partial h_{ij}} - F b^{ki} b^{jk})\bar{R}_{\nu imj}\bar{g}(\lambda\partial_r, e_m) \\
& - F b^{kp} b^{qk} \frac{\partial F}{\partial h_{ij}} (\bar{R}_{\nu ipj;q} + \bar{R}_{\nu pqi;j} + h_{mi}\bar{R}_{mpqj} + h_{pm}\bar{R}_{miqj}).
\end{aligned}$$

When N is a real space form

If the sectional curvature of N is constant ϵ, then $\bar{R}_{\nu ijk} = 0$ and

$$\bar{R}_{ijkl} = \epsilon(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

Thus

$$\begin{aligned}\mathcal{L}Z = {} & 2\frac{\partial F}{\partial h_{ij}}\nabla_i F\nabla_j \mathrm{tr}\, b + \bar{g}(\lambda\partial_r, \nabla(F\mathrm{tr}\, b)) + (\beta - 1)\lambda'(F\mathrm{tr}\, b - \frac{n}{\beta}\sum_i \frac{\partial F}{\partial h_{ii}}) \\ & + C(\frac{\partial F}{\partial h_{ij}} h_{il}h_{jl}\mathrm{tr}\, b - \beta n F) + Fb^{kp}b^{qk}\frac{\partial^2 F}{\partial h_{ij}\partial h_{st}} h_{ijp}h_{stq} \\ & + 2Fb^{ks}b^{pt}b^{kq}\frac{\partial F}{\partial h_{ij}} h_{sti}h_{pqj} + \epsilon F(\beta F\mathrm{tr}(b^2) - \mathrm{tr}\, b\sum_i \frac{\partial F}{\partial h_{ii}}).\end{aligned}$$

λ'-term, C-term and ϵ-term

We pick the λ'-term, C-term and ϵ-term in $\mathcal{L}Z$ as follows:

Lemma 3.1.

$$(\beta-1)\lambda'(F\mathrm{tr}b-\frac{n}{\beta}\sum_i\frac{\partial F}{\partial h_{ii}})=\frac{(\beta-1)}{\beta}\sum_{i>j}\frac{\lambda'}{\kappa_i\kappa_j}(\frac{\partial F}{\partial\kappa_i}\kappa_i-\frac{\partial F}{\partial\kappa_j}\kappa_j)(\kappa_i-\kappa_j),$$

$$C(\frac{\partial F}{\partial h_{ij}}h_{il}h_{jl}\mathrm{tr}b-\beta nF)=C\sum_{i>j}\frac{1}{\kappa_i\kappa_j}\Big(\frac{\partial F}{\partial\kappa_i}\kappa_i^2-\frac{\partial F}{\partial\kappa_j}\kappa_j^2\Big)(\kappa_i-\kappa_j),$$

$$\epsilon F(\beta F\mathrm{tr}(b^2)-\mathrm{tr}b\sum_i\frac{\partial F}{\partial h_{ii}})=\epsilon F\sum_{i>j}\frac{1}{\kappa_i^2\kappa_j^2}(\frac{\partial F}{\partial\kappa_i}\kappa_i^2-\frac{\partial F}{\partial\kappa_j}\kappa_j^2)(\kappa_i-\kappa_j).$$

- Under our assumption, these three terms are all non-negative;
- And each "=" holds iff $\kappa_1=\cdots=\kappa_n$.

Estimate of Z at a umbilic point if $\beta > 1$

For any function F satisfying Condition $(*)$,

$$\mathcal{L}Z + R(\nabla Z) \geq \lambda'\text{-term} + C\text{-term} + \epsilon\text{-term} + Q_i(\kappa)(\nabla_i F)^2,$$

where

$$Q_i(\kappa) = 2\frac{\partial \log F}{\partial \kappa_i}(n\kappa_i^{-1} - \mathrm{tr}\, b) - \frac{2n}{\beta}\kappa_i^{-1}\frac{\partial \log F}{\partial \kappa_i} + \frac{(n+1)\beta - n + 1}{\beta}\kappa_i^{-2}.$$

If at $\bar{x}$, $\kappa_1 = \cdots = \kappa_n$, then

$$Q_i = \frac{(n-1)(\beta-1)}{\beta}\kappa_i^{-2} > 0, \quad \text{if } \beta > 1.$$

Introduce an auxiliary test function W

Define W by

$$W := \frac{F}{\kappa_{\min}} - \frac{\beta - 1}{\beta}\Phi.$$

Since

$$\mathrm{tr}\, b = \kappa_1^{-1} + \kappa_2^{-1} + ... + \kappa_n^{-1},$$

$$nW \geq Z,$$

and "=" holds iff

$$\kappa_1 = \kappa_2 = \cdots = \kappa_n.$$

Analysis at a maximum point of W

At a maximum point of W, $\mathcal{L}W \le 0$ and $\nabla W = 0$.

Lemma.

Any maximum point $\bar{x}$ of W is a umbilic point of M and $\nabla F(\bar{x}) = 0$,

- *for $\beta > 1$ and $C \ge 0$ or $\beta \ge 1$ and $C > 0$, when $N = \mathbb{R}^{n+1}$;*
- *for $\beta \ge 1$ and $C \ge 0$, when $N = \mathbb{S}^{n+1}_{+}$.*

Maximum points of W and Z

$$Z = F\,\mathrm{tr}\,b - \frac{n(\beta-1)}{\beta}\Phi, \qquad W := \frac{F}{\kappa_{\min}} - \frac{\beta-1}{\beta}\Phi.$$

$Z \le nW$ everywhere and $Z = nW$ at umbilic points.

- If W attains its maximum at $\bar{x}$, then $\kappa_1(\bar{x}) = \cdots = \kappa_n(\bar{x})$.
- This implies $\forall x \in M$, $Z(\bar{x}) = nW(\bar{x}) \ge nW(x) \ge Z(x)$, i.e.,$\bar{x}$ is also a maximum point of Z.
- By the analysis of $\mathcal{L}Z$ at a umbilic point, there exists a neighborhood U of $\bar{x}$ s.t. $\mathcal{L}Z + R(\nabla Z) \ge 0$. Since $Z(\bar{x}) = Z_{\max}$, by the strong maximum principle, Z is constant in U.
- Since $Z(\bar{x}) \ge nW \ge Z(x)$, W is also constant in U. Hence, the set of maximal points of W is an open set.

Due to the connectedness of M, W is a constant on M.

M is umbilic

Since W is constant, by the argument of W at its maximum point, M is totally umbilic.

From

$$0 = \nabla F = \kappa_i \bar{g}(\lambda \partial_r, e_i) e_i,$$

we see ∂_r is a normal direction of M and M is a slice in the warped product.

$N = \mathbb{R}^{n+1}$ and α small

For $F = \sigma_k^\alpha$, if $2 \le k \le n$ and $\frac{n-1}{kn+k-2} \le \alpha \le \frac{1}{2}$, then

$$\mathcal{L}Z + R(\nabla Z) \ge 0.$$

By the strong maximum principle, we know Z is a constant, this implies the λ'-term $= 0$.

- When $2 \le k \le n-1$, λ'-term $= 0 \Rightarrow M$ is totally umbilic.
- When $k = n$, λ'-term $\equiv 0$. If $\frac{1}{n+2} < \alpha \le \frac{1}{2}$, then $\nabla F = 0$, thus M is a round sphere centered at the origin; if $\alpha = \frac{1}{n+2}$,

$$C_{ijk} = \frac{1}{2}K^{-\frac{1}{n+2}}h_{ijk} + \frac{1}{2}h_{jk}\nabla_i K^{-\frac{1}{n+2}} + \frac{1}{2}h_{ki}\nabla_j K^{-\frac{1}{n+2}} + \frac{1}{2}h_{ij}\nabla_k K^{-\frac{1}{n+2}} = 0.$$

In $\mathbb{R}^{n+1}$, by Berwald-Pick Theorem, M is an ellipsoid.

Jellet-Liebmann theorem

- Jellet-Liebmann theorem: Any closed star-shaped (or convex) immersed hypersurface in Euclidean space with constant mean curvature is a round sphere.
- Montiel (1999): generalized to a class of warped products. The key tool is the Minkowski type integral formulae.
- Alias-Impera-Rigoli (2013), Brendle-Eichmair (2014), Wu-Xia (2014): hypersurfaces with constant higher order mean curvature or Weingarten hypersurfaces in warped products

Alexandrov theorem

- Alexandrov theorem: Any closed embedded hypersurface of constant mean curvature in Euclidean space is a round sphere.
- Brendle (2013): generalized to a class of warped product manifolds. The key steps are the Minkowski type formula and a Heintze-Karcher type inequality.
- Brendle-Eichmair (2014), Wu-Xia(2014): This also works for Weingarten hypersurfaces.
- Kwong-Lee-Pyo proved Alexandrov type results for closed embedded hypersurfaces with radially symmetric higher order mean curvature in a class of warped products.

Theorem (Brendle-Eichmair 2014)

Let M be a closed, embedded hypersurface in the deSitter-Schwarzschild manifold that is star-shaped and convex. Moreover, suppose that $\sigma_k =$ constant. Then M is a slice.

An integral formula

Lemma.

Suppose $x(M)$ is a closed hypersurface of $\bar{M}$. The following equality holds

$$\int_M \{ -(n-k)\langle \nabla\sigma_k, \lambda\partial_r\rangle + ((n-k)\sigma_1\sigma_k - n(k+1)\sigma_{k+1})u$$
$$- n\bar{R}_{\nu pjp}\langle \lambda\partial_r, e_j\rangle \sigma_{k-1;jp}\} d\mu = 0.$$

Proof.

$$\Phi := x^*(\int_0^r \lambda(s)\,ds), \qquad u := \langle \lambda\partial_r, \nu\rangle.$$

To calculate $\nabla_i(k\sigma_k\nabla_i\Phi - n\frac{\partial\sigma_k}{\partial h_{ij}}\nabla_j u)$ and use the divergence theorem.

Jellet-Liebmann type theorem-I

Theorem 4.1 (Gao-M. 2019).

Suppose that $(\bar{M}^{n+1} = [0, \bar{r}) \times_\lambda P^n, \bar{g} = dr^2 + \lambda^2(r) g^P)$ $(0 < \bar{r} \leq \infty)$ *is a warped product manifold satisfying*

$$Ric^P \geq (n-1)(\lambda'^2 - \lambda\lambda'') g^P,$$

and $x : M \to \bar{M}$ *is an immersion of a closed orientable hypersurface* M^n *in* $\bar{M}$*. If* $x(M)$ *is star-shaped and satisfies*

$$\langle \nabla H, \partial_r \rangle \leq 0, \tag{4.1}$$

then $x(M)$ *must be totally umbilic.*

Remark. If M has constant mean curvature, Theorem 4.1 reduces to the Jellett-Liebmann type theorem proved by Montiel (1999).

Jellet-Liebmann type theorem-II

Corollary (Gao-M. 2019).

Under the same assumption of Theorem 4.1, if $x(M)$ is star-shaped and satisfies

$$H = \phi(r),$$

where $\phi(r) = x^*(\Phi(r))$ and $\Phi(r)$ is a positive non-increasing function of r, then $x(M)$ must be totally umbilic.

Corollary (Gao-M. 2019).

Under the same assumption of Theorem 4.1, if $x(M)$ is strictly convex and satisfies

$$H^{-\alpha} = \langle \lambda \partial_r, \nu \rangle,$$

where $\alpha > 0$ is a constant, then $x(M)$ is a slice $\{r_0\} \times P$ for some $r_0 \in (0, \bar{r})$.

Jellet-Liebmann type theorem-III

Theorem 4.2 (Gao-M. 2019).

Suppose that $\bar{M}^{n+1} = [0, \bar{r}) \times_\lambda P^n$ is a warped product manifold, where (P, g^P) is a closed Riemannian manifold with constant sectional curvature ϵ and

$$\frac{\lambda(r)''}{\lambda(r)} + \frac{\epsilon - \lambda(r)'^2}{\lambda(r)^2} \geq 0. \tag{4.2}$$

Let $x : M \to \bar{M}$ be an immersion of a closed orientable hypersurface M^n in $\bar{M}$. For any fixed k with $2 \leq k \leq n-1$, if $x(M)$ is k-convex, star-shaped and satisfies

$$\langle \nabla H_k, \partial_r \rangle \leq 0,$$

then $x(M)$ must be totally umbilic.

Jellet-Liebmann type theorem-IV

Corollary (Gao-M. 2019).

Suppose that $\bar{M}^{n+1} = [0, \bar{r}) \times_\lambda P^n$ is a warped product manifold, where (P, g^P) is a closed Riemannian manifold with constant sectional curvature ϵ and

$$\frac{\lambda(r)''}{\lambda(r)} + \frac{\epsilon - \lambda(r)'^2}{\lambda(r)^2} > 0. \tag{4.3}$$

Let $x : M \to \bar{M}$ be an immersion of a closed orientable hypersurface M^n in $\bar{M}$. For any fixed k with $2 \leq k \leq n-1$, if $x(M)$ is k-convex, star-shaped and $H_k =$ constant, then $x(M)$ is a slice $\{r_0\} \times P$ for some $r_0 \in (0, \bar{r})$.

This implies that the embeddedness condition in Brendle-Eichmair's result is not necessary.

Jellet-Liebmann type theorem-V

Corollary (Gao-M. 2019).

Under the same assumption of Theorem 4.2, if for any fixed k with $2 \leq k \leq n-1$, $x(M)$ is k-convex, star-shaped and satisfies

$$H_k = \phi(r),$$

where $\phi(r) = x^*(\Phi(r))$ and $\Phi(r)$ is a positive non-increasing function of r, then $x(M)$ must be totally umbilic.

Corollary (Gao-M. 2019).

Under the same assumption of Theorem 4.2, if for any fixed k with $2 \leq k \leq n-1$, $x(M)$ is strictly convex and satisfies

$$H_k^{-\alpha} = \langle \lambda \partial_r, \nu \rangle,$$

where $\alpha > 0$ is a constant, then $x(M)$ is a slice $\{r_0\} \times P$ for some $r_0 \in (0, \bar{r})$.

Theorem (Gao-M. 2019)

Suppose that $(\bar{M}, \bar{g})$ is a warped product manifold satisfying conditions (C1)-(C4). Let $x : M \to \bar{M}$ be an immersion of a connected closed embedded orientable hypersurface M^n in $\bar{M}$. If $H_k > 0$ and $x(M)$ satisfies

$$H_k^{-\alpha}\lambda' = \langle \lambda \partial_r, \nu \rangle, \tag{4.4}$$

for any fixed k with $1 \leq k \leq n$ and $\alpha \geq \frac{1}{k}$, then $x(M)$ is a slice $\{r_0\} \times P$ for some $r_0 \in (0, \bar{r})$.

Further questions

1. What is the asymptotic behavior of the σ_k^α-curvature flow?More explicitly, does it asymptotic to a σ_k^α self-similar solution?
2. Does σ_k^α-self-similar solution have a variational characterization description?
3. Does the result hold if *strictly convexity* is changed to *k-convexity*?
4. Existence of non-compact complete (k-)convex graph solution to σ_k^α-flow?

Thanks for your attention!