Regularity of free boundary in optimal transportation

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Overview

1. Optimal partial transport
   - Free boundary problem, known results
   - Higher order regularity of free boundary, main ideas

2. Optimal transport with target consists of two disjoint parts
   - Lipschitz, $C^{1,\alpha}$ regularity
   - Higher order regularity

Optimal partial transport

Let Ω, Λ be two disjoint, convex domains associated with densities f and g respectively. Let $c : \mathbb{R}^n \times \mathbb{R}^n$ be the cost function. Let m be a positive number satisfying

$$m \leq \min\{\int_\Omega f, \int_\Lambda g\}.$$

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- This problem has been studied by Caffarelli and McCann [*Ann. of Math.* 2010], Figalli [ARMA, 2010], Indrei [JFA 2013].
- For the standard optimal transport problem $m = \int_\Omega f = \int_\Lambda g$, regularity issue has been studied by many experts during the last decades, to list a few: Delanoe, Urbas, Caffarelli, Ma, Trudinger, Wang, Loeper, Vilanni, Liu, Li, Santambrogio, Kim, Figalli, McCann, Kitagawa, Guillen.....

Optimal plan

A transport plan is described as a non-negative, finite Borel measure γ on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\gamma(A \times \mathbb{R}^n) \leq \int_A f(x)dx, \quad \gamma(\mathbb{R}^n \times A) \leq \int_A g(x)dx$$

for any Borel set A. An optimal transport plan minimises the following functional

$$\gamma \mapsto \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x,y)d\gamma(x,y). \tag{1}$$

Optimal map, free boundary

- Caffarelli and McCann proved that γ_m, the minimiser of of (1), is characterised by

$$\gamma_m := (Id \times T_m)_{\#} f_m = (T_m^{-1} \times Id)_{\#} g_m,$$

where T_m is the optimal transport map between active regions $U \subset \Omega$ and $V \subset \Omega$, $f_m = f\chi_U$, and $g_m = g\chi_V$.

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- Indeed $T_m = Du$ for some convex function u solving

$$(Du)_\#(f_m + (g - g_m)) = g. \tag{2}$$

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- The free boundary in Ω (resp. Λ) is defined as $\partial U \cap \Omega$ (resp. $\partial V \cap \Lambda$).

A formulation of double obstacle problem

Caffarelli and McCann formulated the optimal partial transport problem as a double obstacle problem for Monge-Ampère equation as following. Given positive f, g supported in X, Y respectively. Find a convex function u such that

$$g(Du(x))\det(D^2u(x)) = f(x) \text{ on } U := \{x : u(x) > \frac{|x|^2}{2}\} \tag{3}$$

with boundary conditions

$$Du(U) \subset V := \{x : u^*(x) > \frac{|x|^2}{2}\} \text{ and } \int_U f = \int_V g,$$

where u^* is the Legendre transform of u.

Boundary $C^{1,\alpha}$ regularity of optimal transport problem

Caffarelli and McCann's $C^{1,\alpha}$ regularity of the free boundary is based on the method used in proving the following theorem.

Theorem (Caffarelli 92)

Let Ω, Λ be two convex domains associated with densities $\frac{1}{\lambda} < f, g < \lambda$ respectively. Suppose u is the convex function solving $(\partial u)_\sharp f\chi_\Omega = g\chi_\Lambda$. Then, $u \in C^{1,\alpha}(\bar{\Omega})$.

Remark. Instead of proving the $C^{1,\alpha}$ regularity of u directly, Caffarelli first showed that u^*, the Legendre transform of u, has some quantitative strict convexity, and then by duality the regularity of u follows.

Known results

- Caffarelli and McCann proved the existence and uniqueness of solutions to the optimal partial transport problem, they showed that that the free boundary $\partial U \cap \Omega$ is $C^{1,\alpha}$ under the condition that Ω and Λ are convex and disjoint.

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- The last result was later improved by Indrei to a locally $C^{1,\alpha}$ regularity result away from the common region and up to a relatively closed singular set
- Higher order regularity of free boundary is a very difficult open problem.

Main result for optimal partial transport problem

Our main result is a "Flatness implies smoothness" type theorem, and the flatness is guaranteed by the assumption that $dist(\Omega, \Lambda)$ is sufficiently large.

Theorem (C-Liu, 2018)

Given two bounded, C^2, uniformly convex domains Ω, Λ associated with positive densities f and g. Given mass m to be transported. Suppose U is the active region of Ω. Then, for any $\delta > 0$,

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- *a) if f, g are continuous, then there exists a constant $L > 0$ such that $\partial U \cap \Omega_\delta$ is $C^{1,\beta}$ for any $\beta \in (0, 1)$, provided $dist(\Omega, \Lambda) \geq L$, where $\Omega_\delta := \{x \in \Omega : dist(x, \partial\Omega) > \delta\}$.*

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- *b) if f, g are C^α, then there exists a constant $L > 0$ such that $\partial U \cap \Omega_\delta$ is $C^{2,\alpha'}$ for some $0 < \alpha' < \alpha$, provided $dist(\Omega, \Lambda) \geq L$.*

Observations

- Caffarelli and McCann proved that the unit inner normal of $\partial U \cap \Omega$, is given by
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- Fix any $x_0 \in \Omega, y_0 \in \Lambda$, we have that $\frac{x-y}{|x-y|}$ is uniformly close to some unit vector $e := \frac{x_0 - y_0}{|x_0 - y_0|}$ for any $x \in \Omega, y \in \Lambda$, provided $dist(\Omega, \Lambda)$ is sufficiently large. Hence $|\nu(x) - e|$ can be as small as we want provided $dist(\Omega, \Lambda)$ is large enough.

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- By rotating and translating the coordinates, we may assume
$$\{x^n > \delta\} \cap \Omega \subset U \subset \{x^n > -\delta\} \cap \Omega,$$
and
$$\{y^n < -\delta\} \cap \Lambda \subset V \subset \{y^n < \delta\} \cap \Lambda,$$
where $\delta \to 0$ as $dist(\Omega, \Lambda) \to \infty$.

Perturbation

Denote by $\tilde{u}$ the potential function of optimal transport between U and V.

- Let $U_\infty := \{x^n > 0\} \cap \Omega$, $V_\infty := \{y^n < 0\} \cap \Lambda$. Let v be the convex function solving $(Dv)_\sharp \tilde{f} \chi_{U_\infty} = g \chi_{V_\infty}$, with $v(x_0) = \tilde{u}(x_0)$ for some $x_0 \in U_\infty$.

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- Then, by a standard compactness argument we have that $\tilde{u}$ can be as close to v in L^∞ norm as we want provided $dist(\Omega, \Lambda)$ is large enough.

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- Then, by a standard compactness argument we have that $\tilde{u}$ can be as close to v in L^∞ norm as we want provided $dist(\Omega, \Lambda)$ is large enough.
- If v is $C^{2,\alpha}$, by some sort of perturbation argument ($C^{2,\alpha}$ means close to a quadratic, while quadratic is quite stable) we can expect that $\tilde{u}$ should be $C^{2,\alpha'}$ for some $\alpha' < \alpha$.

Regularity of the limit problem

First, lets recall an important result by Caffarelli. The smooth version of the following theorem was also proved by Urbas independently.

Theorem (Caffarelli, 96)

Let u be the potential function of the optimal transport problem from (X, f) to (Y, g), where X, Y are C^2, uniformly convex domains, f, g are positive C^α densities. Then $u \in C^{2,\alpha'}(\bar{X})$ for some $0 < \alpha' < \alpha$.

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- We can not apply Caffarelli's result to out limit problem directly, since our domains U_∞, V_∞ have singular part and flat part. But, by examining his proof of the above theorem carefully, one can find that the key estimates required in the proof can be established in our situation. Therefore we have the following result.

Regularity of the limit problem

Lemma

Suppose U_∞, V_∞ are given above. Suppose f, g are positive densities. Let v be the convex function solves $(Dv)_\sharp f\chi_{U_\infty} = g\chi_{V_\infty}$. Then,
a) if f, g are continuous, then $v \in C^{1,\beta}(U_\infty \cap \Omega_\delta)$ for any $\beta \in (0,1)$.
b) if f, g are C^α, then $v \in C^{2,\alpha'}(U_\infty \cap \Omega_\delta)$ for some $\alpha' \in (0,\alpha)$.

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- Let v^* be the standard Legendre transform of v, and Dv^* is the optimal transport map from V_∞ to U_∞. Then, we also have similar result for v^*.
- This result tells us that around $0 \in \partial U_\infty$, $\tilde{u} \approx \frac{1}{2}|x|^2$ up to an affine transformation. By renormalisation and some delicate estimates, we can show that the conditions of the following result are satisfied, provided $dist(\Omega, \Lambda)$ is sufficiently large.

Lemma (C, Figalli, 2015)

Let $\mathcal{C}_1, \mathcal{C}_2$ be bounded open set satisfying

$$B_{1/3} \cap \{x^n \geq P(x')\} \subset \mathcal{C}_1 \subset B_3 \cap \{x^n \geq P(x')\}$$

$$B_{1/3} \cap \cap\{y^n \geq Q(y')\} \subset \mathcal{C}_2 \subset B_3 \cap \{y^n \geq Q(y')\}.$$

Suppose $f \in C^\alpha(\mathcal{C}_1)$, $g \in C^\alpha(\mathcal{C}_2)$, and $(Du)_\sharp f = g$. There exist small constants $\eta_1 \leq \eta_0$ and $\delta_1 \leq \delta_0$ such that, if

$$\|P\|_{C^2} + \|Q\|_{C^{1,\alpha}} \leq \delta_1, \ \|f - \mathbf{1}\|_{L^\infty(\mathcal{C}_1)} + \|g - \mathbf{1}\|_{L^\infty(\mathcal{C}_2)} \leq \delta_1,$$

and

$$\left\| u - \frac{1}{2}|x|^2 \right\|_{L^\infty(\mathcal{C}_1)} \leq \eta_1, \tag{4}$$

then, there exists $\rho_2 > 0$ small such that $u \in C^{2,\alpha}_{\mathrm{loc}}(\mathcal{C}_1 \cap B_{\rho_2}) \cap C^{2,\alpha'}(\overline{\mathcal{C}_1 \cap B_{\rho_2}})$ for some $\alpha' \in (0, \alpha)$.

Optimal transport with target consists of two disjoint parts

- We consider optimal transportation from a source domain U associated with density f to the target $V = V_1 \cup V_2$ associated with density g, where V_1 and V_2 are two domains separated by a hyperplane H.

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- Denote by u (resp. v) the convex function solving $(\partial u)_\sharp f\chi_U = g\chi_V$ (resp. $(\partial v)_\sharp g\chi_V = f\chi_U$). In the following, we assume $1/\lambda < f, g < \lambda$ for some positive constant λ.

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- The more general model (target consists of many disjoint parts) is also independently studied by McCann and Kitagawa. In particular, they investigate the $C^{1,\alpha}$ regularity of the free boundary.

Free boundary: Lipschitz, $C^{1,\alpha}$ regularity

First, under mild conditions, we have the Lipschitz regularity of the free boundary.

Theorem (C-Liu, 2016)

The interior of $U_1 := \partial v(V_1)$ and $U_2 := \partial v(U_2)$ are disjoint and separated by a Lipschitz hypersurface.

$F := \partial U_1 \cap U$ is the free boundary in our model.

Theorem (C-Liu, 2016)

Suppose U, V_1 and V_2 are strictly convex, then the free boundary $F := \partial U_1 \cap U$ is $C^{1,\alpha}$.

Proof of the Lipschitz regularity

- Assume that $H = \{x^n = 0\} \subset \mathbb{R}^n$, $V_1 \subset \{x^n < 0\}$ and $V_2 \subset \{x^n > 0\}$.

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- Define the cone $\mathcal{C} := \{z | \frac{z}{|z|} \cdot (-e_n) \geq \beta\}$, where $1 > \beta > \sqrt{1 - \alpha^2}$ is a constant. We also denote $\mathcal{C}_x := \{x + z | z \in \mathcal{C}\}$. A straightforward computation shows that $z_1 \cdot z_2 < 0$ for $z_1 \in D, z_2 \in \mathcal{C}$.

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- On one hand, given any $x \in U_1$, for any $\tilde{x} \in \mathcal{C}_x \cap U$, we have that

$$(\tilde{x} - x) \cdot (y_2 - y_1) < 0$$

for any $y_2 \in V_2$, where $y_1 \in \partial u(x)$.

Proof continued

- On the other hand, by monotonicity of convex function we have

$$(\tilde{x} - x) \cdot (z - y_1) \geq 0$$

for any $z \in \partial u(\tilde{x})$. Therefore $\partial u(\tilde{x}) \cap V_2 = \emptyset$.

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- Hence $\partial u(\mathcal{C}_x) \cap V_2 = \emptyset$, which implies $\mathcal{C}_x \subset U_1$. Therefore, we have the characterisation $U_1 = \cup_{x \in U_1} \mathcal{C}_x$.

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- Denote by f_x the Lipschitz function over $\{x^n = 0\}$ with graph $\partial \mathcal{C}_x$. Let $f := \sup_{x \in U_1} f_x$. Since f_x has uniform Lipschitz bound, we have that f is also a Lipschitz function.

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- Then we can also write

$$U_1 := \{x^n \leq f(x^1, \cdots, x^{n-1})\} \cap U.$$

Characterization of the unit normal of the free boundary

- Denote by u_i the restriction of u to $U_i, i = 1, 2$. Note that $u_1 = u_2$ on F. Then, we extend the potential u_i to $\mathbb{R}^n$ in the following way

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  for $i = 1, 2$.
- We can show that $\tilde{u}_i, i = 1, 2$ are C^1, and by implicit function theorem we have that $F = \{\tilde{u}_1 = \tilde{u}_2\} \cap U$ is C^1. Moreover, the unit normal of F at x is given by

  $$\nu(x) = \frac{Du_1(x) - Du_2(x)}{|Du_1(x) - Du_2(x)|}.$$

Higher regularity

Observe that when $dist(V_1, V_2)$ is sufficiently large, $\frac{Du_1(x)-Du_2(x)}{|Du_1(x)D-Du_2(x)|}$ is uniformly close to some unit vector e for any $x \in F$. Hence the free boundary is close to a hyperplane (as close as we want, provided $dist(V_1, V_2)$ is large enough). Then, we can follow our argument for the optimal partial transport problem to establish the following theorem.

Theorem (C-Liu, 2016)

Let U, V_1, V_2, F be as above. Then, given any $\delta > 0$,
a) if f, g are continuous, then there exists a constant $L > 0$ such that $F \cap U_\delta$ is $C^{1,\beta}$ for any $\beta \in (0,1)$, provided $dist(V_1, V_2) \geq L$, where $U_\delta := \{x \in U : dist(x, \partial U) > \delta\}$.
b) if f, g are C^α, then there exists a constant $L > 0$ such that $F \cap U_\delta$ is $C^{2,\alpha'}$ for some $0 < \alpha' < \alpha$ provided $dist(V_1, V_2) \geq L$.

Thanks for your attention