# 量子系统的拓扑和几何
# (Geometric Phases for the observable)

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# Dirac-von Neumann formulation of quantum mechanics

## Mathematical formulation of quantum mechanics

$(P_1)$ A quantum system $Q$ is mathematically associated with a Hilbert space $\mathbb{H}$ and is completely described by a unit vector $\psi$ in $\mathbb{H}$, which is called a (vector) state; and every observable for $Q$ is represented by a self-adjoint operator $A$ on $\mathbb{H}$.

$(P_2)$ The system $Q$ as described by vector states is changed with time according to Schrödinger equation

$$\mathrm{i}\frac{d\psi}{dt} = H\psi, \tag{1.1}$$

where $H$ is a Hermitian operator on $\mathbb{H}$, and $\mathrm{i} = \sqrt{-1}$.

$(P_3)$ At the state $\psi$, by the observation of a quantity described by a self-adjoint operator $A$ with eigenstates $\psi_1, \psi_2, \ldots,$ the state $\psi$ will be changed to the state $\psi_j$ with probability $|\langle\psi, \psi_j\rangle|^2$, whose expectation equals to $\langle\psi, A\psi\rangle$.

# The geometric phase for the quantum state

At a (pure) state $\psi \in \mathbb{H}$, the expectation of any observable $A$,

$$\langle A\rangle_\psi = \langle\psi, A\psi\rangle = \langle\tilde{\psi}, A\tilde{\psi}\rangle, \quad \tilde{\psi} = e^{\mathrm{i}\theta}\psi, \theta \in [0, 2\pi),$$

$e^{\mathrm{i}\theta}$ is called a phase factor and $\theta$ is a phase.

**Berry's Phase**

Berry showed that when $H$ undergoes adiabatic evolution along a closed curve $\Gamma$ in the parameter space $M$, then a state that remains as an eigenstate of $H(R)$ corresponding to a non-degenerate eigenvalue $E(R)$ develops a geometrical phase $\gamma$ which depends only on the geometry of $\Gamma$, as described by Simon.


[Berry] M. V. Berry, Quantal phase factors accompanying adiabatic changes, *Proceedings of the Royal Society of London, Series A* **392** (1984), 45-57.

[Simon] B. Simon, Holonomy, the quantum adiabatic theorem, and Berry's phase, *Physical Review Letters* **51** (1983), 2167-2170.

# The geometric phase for the quantum state

Given a cyclic evolution $C : [0,T] \ni t \to \psi(t)$ with $\psi(T) = e^{i\phi}\psi(0)$, which satisfies the Schrödinger equation

$$\mathrm{i}\frac{d\psi}{dt} = h(t)\psi,$$

where $h(t)$ is a time-dependent Hamiltonian, the geometric phase of Aharonov and Anandan is defined to be

$$\beta = \phi + \int_0^T \langle \psi(s), h(s)\psi(s)\rangle ds$$

depends only on the closed curve $\mathcal{C} : t \to \varrho(t) = |\psi(t)\rangle\langle\psi(t)|$ in the state space satisfying the Liouville-von Neumann equation

$$\mathrm{i}\frac{d\varrho(t)}{dt} = [h(t), \varrho(t)].$$

[AA] Y. Aharonov, J. Anandan, Phase change during a cyclic quantum evolution, *Physical Review Letters* **58** (1987), 1593-1596.

# The fiber bundle over the state space

The principal fiber bundle:

(a) The total space is the unit sphere of $\mathbb{H}$,
$\mathbf{S}(\mathbb{H}) = \{\psi : \|\psi\| = 1, \psi \in \mathbb{H}\}$.

(b) The base space is the state space,
$\mathcal{S}(\mathbb{H}) = \{|\psi\rangle\langle\psi| : \psi \in \mathbf{S}(\mathbb{H})\} \cong \mathcal{P}(\mathbb{H}) = \{[\psi] : \psi \in \mathbf{S}(\mathbb{H})\}$
with $[\psi] = \{e^{\mathrm{i}\theta}\psi : \theta \in [0, 2\pi)\}$.

(c) $\eta = (\mathbf{S}(\mathbb{H}), \mathcal{P}(\mathbb{H}), \pi, \mathcal{U}(1))$ with the structure group $\mathcal{U}(1)$.

(d) The canonical connection $\mathcal{A} = \langle\psi, d\psi\rangle$.

# Expression for the geometric phase on the state space

The curve

$$\tilde{C} : [0, T] \ni t \mapsto \tilde{\psi}(t) = e^{\mathrm{i} \int_0^t \langle \psi(s), h(s)\psi(s) \rangle ds} \psi(t)$$

is the horizontal lift of $\mathcal{C} : t \to \varrho(t) = |\psi(t)\rangle\langle\psi(t)|$ in $\mathcal{S}(\mathbb{H})$, i.e.

$$\langle \tilde{\psi}(t), \frac{d}{dt}\tilde{\psi}(t) \rangle = 0,$$

such that

$$\tilde{\psi}(T) = e^{\mathrm{i}\beta} |\psi(0).$$

Moreover,

$$\beta = \int_0^T \langle \bar{\psi}(t), \frac{d}{dt}\bar{\psi}(t) \rangle dt = \oint_{\bar{C}} \langle \bar{\psi}, d\bar{\psi} \rangle$$

for any closed curve $\bar{C} : [0, T] \ni t \to \bar{\psi}(t)$ with $\bar{\psi}(0) = \bar{\psi}(T)$ such that $\pi(\bar{\psi}) = |\psi\rangle\langle\psi|$.

# New formulation of quantum mechanics

$(Q_1)$ A quantum system $Q$ associated with a Hilbert space $\mathbb{H}$ is described by an orthonormal basis $W$, which is called a prototype; and every observable $W$ is represented by a self-adjoint operator $A$ on $\mathbb{H}$.

$(Q_2)$ The system $Q$ as described by prototypes is changed with time according to the equation for the associated bases $W_t = (\psi_n(t))_{n\geq 1}$ in $\mathbb{H}$,

$$\mathrm{i}\frac{d\psi_n(t)}{dt} = -H\psi_n(t), \quad \forall n \geq 1,$$

where $H$ is a Hermitian operator on $\mathbb{H}$.

$(Q_3)$ For a given prototype $W$, a quantum state is defined as a valuation for the complete family of obeservables which is diagonal under the corresponding basis $W$, which is characterized by either a vector or singular state.

# Cyclic evolution for the observable

- Consider $C : [0, T] \ni t \mapsto W_t = (|\psi_n(t)\rangle)_{n\geq 1}$ satisfying the skew Schrödinger equation

$$\mathrm{i}\frac{d\psi_n(t)}{dt} = -h(t)\psi_n(t), \quad \forall n \geq 1,$$

  with $W_0 = (|\psi_n(0)\rangle)_{n\geq 1}$ being a basis of $\mathbb{H}$.
- Given any fixed family of distinct real numbers $(\lambda_n)_{n\geq 1}$, $X(t) = \sum_{n\geq 1} \lambda_n |\psi_n(t)\rangle\langle\psi_n(t)|$ satisfies the the Heisenberg equation

$$\mathrm{i}\frac{dX(t)}{dt} = [X(t), h(t)]$$

  in the observable space.
- $\mathcal{C} : [0, T] \ni t \mapsto X(t)$ is cyclic in the quantal observable space if $X(0) = X(T)$.

# Geometric phases for the cyclic evolution of observable

1) $C : [0, T] \ni t \mapsto W_t = (|\psi_n(t)\rangle)_{n\geq 1}$ with $|\psi_n(T)\rangle = e^{\mathrm{i}\phi_n}|\psi_n(0)\rangle$ with $\phi_n \in [0, 2\pi)$ for $n \geq 1$.
2) The geometric phases over the $\mathcal{C}$ are defined as

$$\beta_n = \phi_n - \int_0^T \langle\psi_n(t), h(t)\psi_n(t)\rangle dt.$$

3) $\tilde{C} : [0, T] \ni t \mapsto \tilde{W}(t) = (\tilde{\psi}_n(t))_{n\geq 1}$, where

$$\tilde{\psi}_n(t) = e^{-\mathrm{i}\int_0^t \langle\psi_n(s), h(s)\psi_n(s)\rangle ds}\psi_n(t)$$

such that $\tilde{\psi}_n(T) = e^{\mathrm{i}\beta_n}|\psi_n(0)$ and

$$\langle\tilde{\psi}_n(t), \frac{d}{dt}\tilde{\psi}_n(t)\rangle = 0, \forall n \geq 1.$$

# The observable space

- $\mathcal{B}(\mathbb{H})$ is the algebra of all bounded operators on $\mathbb{H}$,
- $\mathcal{O}(\mathbb{H})$ is the set of all self-adjoint operators on $\mathbb{H}$ and, $\mathcal{O}_d(\mathbb{H})$ denotes the subset of $\mathcal{O}(\mathbb{H})$ consisting of those self-adjoint operators with discrete spectrum,
- $\mathcal{U}(\mathbb{H})$ is the group of all unitary operators on $\mathbb{H}$. $I$ always denotes the identity operator on $\mathbb{H}$.
- A complete orthonormal decomposition of the identity operator $I$ in $\mathbb{H}$, is $O = \{|n\rangle\langle n| : n \geq 1\}$ with $\sum_n |n\rangle\langle n| = I$ and $\langle n|m\rangle = 0$ whenever $n \neq m$.

Denote by $\mathcal{W}(\mathbb{H})$ the set of all complete orthonormal decompositions in $\mathbb{H}$. The distance $D_{\mathcal{W}}$ on $\mathcal{W}(\mathbb{H})$ is defined as: For $O, O' \in \mathcal{W}(\mathbb{H})$,

$$D_{\mathcal{W}}(O, O') = \inf\{\|I - U\| : \ U^{-1}O'U = O, \ U \in \mathcal{U}(\mathbb{H})\}.$$

$(\mathcal{W}(\mathbb{H}), D_{\mathcal{W}})$ is called the observable space.

# The topology for the observable space

$\mathcal{G}(\mathbb{N})$ denotes the automorphism group of $\mathbb{N}$. For an arbitrary fixed basis $(|n\rangle)_{n\geq 1}$ of $\mathbb{H}$, $\mathcal{U}(1)^{\mathbb{N}}$ and $\mathcal{G}(\mathbb{N}) \times \mathcal{U}(1)^{\mathbb{N}}$ are represented as

$$\mathcal{U}(1)^{\mathbb{N}} = \Big\{ \sum_{n\geq 1} e^{\mathrm{i}\theta_n} |n\rangle\langle n| : \ \forall \theta_n \in [0, 2\pi) \Big\},$$

$$\mathcal{G}(\mathbb{N}) \times \mathcal{U}(1)^{\mathbb{N}} = \Big\{ \sum_{n\geq 1} e^{\mathrm{i}\theta_n} |\sigma(n)\rangle\langle n| : \ \forall \sigma \in \mathcal{G}(\mathbb{N}), \forall \theta_n \in [0, 2\pi) \Big\}.$$

**Topological space for a quantum system**

Given an arbitrary fixed basis $(|n\rangle)_{n\geq 1}$ of $\mathbb{H}$,

$$\mathcal{W}(\mathbb{H}) \cong \frac{\mathcal{U}(\mathbb{H})}{\mathcal{G}(\mathbb{N}) \times \mathcal{U}(1)^{\mathbb{N}}}.$$

$\mathcal{W}(\mathbb{C}^d)$ is topologically non-trivial as its fundamental group is isomorphic to $\Pi(d)$, the permutation group of $d$ objects.

# The principal fiber bundle

Fix a point $O_0 = \{|n\rangle\langle n| : n \geq 1\} \in \mathcal{W}(\mathbb{H})$.

- For any $O \in \mathcal{W}(\mathbb{H})$, we write

$$\mathcal{F}_O = \{U \in \mathcal{U}(\mathbb{H}) : \; U^\dagger O U = O_0\}.$$

- Define $\mathcal{G}_{O_0} = \{\sum_{n\geq 1} e^{\mathrm{i}\theta_n} |\sigma(n)\rangle\langle n| : \; \forall \sigma \in \mathcal{G}(\mathbb{N}), \forall \theta_n \in [0, 2\pi)\}$, and the action of $\mathcal{G}_{O_0}$ on $\mathcal{F}_O : (G, U) \mapsto UG$ for any $G \in \mathcal{G}_{O_0}$ and for all $\mathcal{F}_O$. Clearly, $\mathcal{G}_{O_0}$ is the structure group of $\mathcal{F}_O$.

Since

$$\mathcal{U}(\mathbb{H}) = \bigcup_{O \in \mathcal{W}(\mathbb{H})} \mathcal{F}_O,$$

the principal fiber bundle over the quantal observable space is the follows:

$$P_{O_0} = (\mathcal{U}(\mathbb{H}), \mathcal{W}(\mathbb{H}), \Pi, \mathcal{G}_{O_0}),$$

where $\Pi^{-1}(O) = \mathcal{F}_O$ for $O \in \mathcal{W}(\mathbb{H})$.

# Tangent space for the structure group

Denote by $\mathcal{Q}(\mathbb{H})$ the set of all densely defined operators in $\mathbb{H}$. Fix $O_0 = \{|n\rangle\langle n| : n \geq 1\} \in \mathcal{W}(\mathbb{H})$, and let $\mathcal{D}(O_0) = \mathrm{span}\{|n\rangle : n \geq 1\}$, which is a densely subspace of $\mathbb{H}$.

## Tangent vectors for the structure group $\mathcal{G}_{O_0}$

Fix $O_0 \in \mathcal{W}(\mathbb{H})$. For a given $U \in \mathcal{G}_{O_0}$, an operator $Q \in \mathcal{Q}(\mathbb{H})$ is called a tangent vector at $U$ for $\mathcal{G}_{O_0}$, if $\mathcal{D}(O_0) \subset \mathrm{Dom}(Q)$ and there is a strongly continuous curve $\chi : (-\varepsilon, \varepsilon) \ni t \mapsto U(t) \in \mathcal{G}_{O_0}$ with $\chi(0) = U$ such that for every $h \in \mathrm{Dom}(Q)$, the limit

$$\lim_{t\to 0} \frac{U(t)(h) - U(h)}{t} = Q(h)$$

in $\mathbb{H}$. In this case, we denote by $Q = \frac{d\chi(t)}{dt}\Big|_{t=0}$. The set of all tangent vectors at $U$ is denoted by $T_U\mathcal{G}_{O_0}$, and $T\mathcal{G}_{O_0} = \bigcup_{U\in\mathcal{G}_{O_0}} T_U\mathcal{G}_{O_0}$.

# Tangent space for the base space

1) Fix $O_0 \in \mathcal{W}(\mathbb{H})$. For a continuous curve $\chi : (a,b) \ni t \mapsto O(t) \in \mathcal{W}(\mathbb{H})$, a subset $\mathcal{A}$ of $\mathcal{Q}(\mathbb{H})$ is called a tangent vector of $\chi$ at a fixed $t_0 \in (a,b)$ relative to $O_0$, if for any $Q \in \mathcal{A}$, $\mathcal{D}(O_0) \subset \mathrm{Dom}(Q)$ and there is a strongly continuous curve $\gamma : (a,b) \ni t \mapsto U_t \in \mathcal{F}_{O(t)}$ such that for every $h \in \mathrm{Dom}(Q)$, the limit
$$\lim_{t\to t_0} \frac{U_t(h) - U_{t_0}(h)}{t - t_0} = Q(h)$$
in $\mathbb{H}$. In this case, we denote by $\mathcal{A} = \frac{dO(t)}{dt}\big|_{t=t_0} = \frac{d\chi(t)}{dt}\big|_{t=t_0}$.
2) Fix $O_0 \in \mathcal{W}(\mathbb{H})$. Given $O \in \mathcal{W}(\mathbb{H})$, a tangent vector of $\mathcal{W}(\mathbb{H})$ at $O$ relative to $O_0$ is define to be a subset $\mathcal{A}$ of $\mathcal{Q}(\mathbb{H})$, provided $\mathcal{A}$ is a tangent vector of some continuous curve $\chi$ at $t=0$, where $\chi : (-\varepsilon, \varepsilon) \ni t \mapsto O(t) \in \mathcal{W}(\mathbb{H})$ with $\chi(0) = O$, i.e., $\mathcal{A} = \frac{dO(t)}{dt}\big|_{t=0}$. We denote by $T_O\mathcal{W}(\mathbb{H})$ the set of all tangent vectors at $O$, and write $T\mathcal{W}(\mathbb{H}) = \bigcup_{O\in\mathcal{W}(\mathbb{H})} T_O\mathcal{W}(\mathbb{H})$.

# Tangent space for the total space

1) Fix $O_0 \in \mathcal{W}(\mathbb{H})$. For a given $P \in \mathcal{U}(\mathbb{H})$, an operator $Q \in \mathcal{Q}(\mathbb{H})$ is called a tangent vector of $P_{O_0}$ at $P$ relative to $O_0$, if $\mathcal{D}(O_0) \subset \mathrm{Dom}(Q)$ and there exists a strongly continuous curve $\gamma : (-\varepsilon, \varepsilon) \ni t \mapsto P_t \in \mathcal{U}(\mathbb{H})$ with $\gamma(0) = P$, such that for any $h \in \mathrm{Dom}(Q)$,
$$\lim_{t\to 0} \frac{P_t(h) - P(h)}{t} = Q(h)$$
in $\mathbb{H}$. In this case, we write $Q = \frac{dP_t}{dt}\Big|_{t=0} = \frac{d\gamma(t)}{dt}\Big|_{t=0}$. Denote by $T_P P_{O_0}(\mathbb{H})$ the set of all tangent vectors of $P_{O_0}$ at $P$ relative to $O_0$, and write $TP_{O_0}(\mathbb{H}) = \bigcup_{P\in P_{O_0}(\mathbb{H})} T_P P_{O_0}(\mathbb{H})$.

2) Given $P \in P_{O_0}(\mathbb{H})$, a tangent vector $Q \in T_P P_{O_0}(\mathbb{H})$ is said to be vertical, if there is a strongly continuous curve $\Gamma : (-\varepsilon, \varepsilon) \ni t \mapsto P_t \in F_{\Pi(P)}$ with $\Gamma(0) = P$ such that $Q = \frac{d\Gamma(t)}{dt}\Big|_{t=0}$. We denote by $V_P P_{O_0}(\mathbb{H})$ the set of all vertical tangent vectors at $P$.

# Connection over the bundle

A connection on the principal fiber bundle $P_{O_0}$ is a family of linear functionals $\Omega = \{\Omega_P : \ P \in P_{O_0}(\mathbb{H})\}$, where for each $P \in P_{O_0}(\mathbb{H})$, $\Omega_P$ is a linear functional in $T_P P_{O_0}(\mathbb{H})$ with values in $T\mathcal{G}_{O_0}$, satisfying the following conditions:

(1) For any $P \in \mathcal{U}(\mathbb{H})$, every vertical tangent vector $Q \in V_P P_{O_0}(\mathbb{H})$ satisfies the equation $\Omega_P(Q) = P^\dagger Q$.

(2) $\Omega_P$ depends continuously on $P$ in a certain topology.

(3) Under the right action of $\mathcal{G}_{O_0}$ on $P_{O_0}(\mathbb{H})$, $\Omega$ transforms according to

$$\Omega_{R_G(P)}[(R_G)_*(Q)] = G^{-1}\Omega_P(Q)G$$

for $G \in \mathcal{G}_{O_0}$, $P \in P_{O_0}(\mathbb{H})$, and $Q \in T_P P_{O_0}(\mathbb{H})$.

Such a connection is simply called an $O_0$-connection.

# The canonical connection

Given a fixed $O_0 \in \mathcal{W}(\mathbb{H})$, we define $\check{\Omega} = \{\check{\Omega}_P : P \in \mathcal{U}(\mathbb{H})\}$ as follows: For each $P \in \mathcal{U}(\mathbb{H})$, $\check{\Omega}_P : T_P P_{O_0}(\mathbb{H}) \mapsto T\mathcal{G}_{O_0}$ is defined by

$$\check{\Omega}_P(Q) = P^\dagger \star Q$$

for any $Q \in T_P P_{O_0}(\mathbb{H})$, where

$$P^\dagger \star Q = \sum_{n \geq 1} \langle n|P^\dagger Q|n\rangle |n\rangle\langle n|.$$

This is clearly an $O_0$-connection on $P_{O_0}$. In this case, we write $\check{\Omega}_P = P^\dagger \star dP$ for any $P \in \mathcal{U}(\mathbb{H})$.

# Quantum lifts

The evolution of an exact cyclic observable is defined to be a closed loop

$$C_W : \ [0,T] \ni t \longmapsto O(t) \in \mathcal{W}(\mathbb{H}), \quad O(0) = O(T),$$

in the base space $\mathcal{W}(\mathbb{H})$.

## Quantum lifts

Fix a point $O_0 = \{|n\rangle\langle n| : n \geq 1\} \in \mathcal{W}(\mathbb{H})$. For a continuous curve $C_W : [0,T] \ni t \longmapsto O(t) \in \mathcal{W}(\mathbb{H})$, a lift of $C_W$ with respect to $O_0$ is defined to be a continuous curve

$$C_P : [0,T] \ni t \longmapsto U(t) \in \mathcal{U}(\mathbb{H})$$

such that $U(t) \in \mathcal{F}_{O(t)}$ for any $t \in [0,T]$, that is, $\{U(t)|n\rangle : \ n \geq 1\} \in Ba(O(t))$ for every $t \in [0,T]$.

# Quantum parallel transport

**Horizontal lift for $C_W$**

Fix $O_0 \in \mathcal{W}(\mathbb{H})$ and let $\Omega$ be an $O_0$-connection on $P_{O_0}(\mathbb{H})$. Let $C_W : [0, T] \ni t \longmapsto O(t) \in \mathcal{W}(\mathbb{H})$ be a continuous curve. A horizontal lift of $C_W$ with respect to $O_0$ is defined to be a $O_0$-lift of $C_W$, $C_P : [0, T] \ni t \longmapsto \tilde{U}(t)$ such that

$$\Omega_{\tilde{U}(t)}\left[\frac{d\tilde{U}(t)}{dt}\right] = 0$$

for every $t \in [0, T]$.

In this case, the curve $t \mapsto \tilde{U}(t)$ is called the parallel transportation along $C_W$ associated with the connection $\Omega$ on $P_{O_0}(\mathbb{H})$.

# Canonical parallel transportation

Fix $O_0 = \{|n\rangle\langle n| : n \geq 1\} \in \mathcal{W}(\mathbb{H})$. Let $(|\psi_n(t)\rangle)_{n\geq 1}$ be a family of bases such that for each $n \geq 1$, $\psi_n(t)$ is continuously differential $\mathbb{H}$-valued function in $[0, T]$. Then

$$C_W : [0, T] \ni t \to O(t) = \{|\psi_n(t)\rangle\langle\psi_n(t)| : n \geq 1\} \in \mathcal{W}(\mathbb{H})$$

is a continuous curve in $\mathcal{W}(\mathbb{H})$. For $0 \leq t \leq T$, define

$$\tilde{U}(t) = \sum_{n\geq 1} |\tilde{\psi}_n(t)\rangle\langle n|, \quad |\tilde{\psi}_n(t)\rangle = e^{-\int_0^t \langle\psi_n(s), \frac{d}{ds}\psi_n(s)\rangle ds}|\psi_n(t)\rangle.$$

Then $\tilde{C}_P : \ [0, T] \ni t \longmapsto \tilde{U}(t) \in \mathcal{F}_{O(t)}$ is a lift of $C_W$ associated with $O_0$ such that

$$\check{\Omega}_{\tilde{U}(t)}\Big[\frac{d\tilde{U}(t)}{dt}\Big] = 0$$

for all $t \in [0, T]$, where $\check{\Omega}$ is the canonical connection defined above. Therefore, $\tilde{C}_P$ is the *horizontal* lift of $C_W$ in the principal bundle $P_{O_0}$ associated with the canonical connection $\check{\Omega}$.

# Local section

## Section

Let $\mathcal{O}$ be an open subset of $\mathcal{W}(\mathbb{H})$. A mapping $s : \mathcal{O} \subset \mathcal{W}(\mathbb{H}) \mapsto \mathcal{U}(\mathbb{H})$ is called a (local) section for $P_{O_0}(\mathbb{H})$, if $s$ is continuous and $s(O) \in \mathcal{F}_O$ for any $O \in \mathcal{O}$. If $\mathcal{O} = \mathcal{W}(\mathbb{H})$, such a section is said to be global.

Let $\Omega$ be an $O_0$-connection on $P_{O_0}(\mathbb{H})$. Let $s : \mathcal{O} \subset \mathcal{W}(\mathbb{H}) \mapsto \mathcal{U}(\mathbb{H})$ be a local section $P_{O_0}(\mathbb{H})$ and $\omega^s_{\mathcal{O}} = \{\omega^s_O : \ O \in \mathcal{O}\}$ be a family of linear functionals on $\bigcup_{O\in\mathcal{O}} T_O\mathcal{W}(\mathbb{H})$ such that for $O \in \mathcal{O}$,

$$\omega^s_{\mathcal{O}}(\mathcal{A}) = \Omega_{s(O)}[s_*(\mathcal{A})],$$

for any $\mathcal{A} \in T_O\mathcal{W}(\mathbb{H})$, where $s_*$ is the pull-forward map of $s$ defined in the usual way. We call $\omega^s_{\mathcal{O}}$ the local connection on $\mathcal{O}$ associated with $\Omega$.

# Locally parallel transportation

If $s'(O) = s(O) \cdot G(O), \quad \forall O \in \mathcal{O}$, we have

$$\omega_O^{s'}(\mathcal{A}) = G(O)^{-1}\omega_O^{s}(\mathcal{A})G(O) + G(O)^{-1}dG(O).$$

## Locally parallel transportation

Fix $O_0 \in \mathcal{W}(\mathbb{H})$ and let $\Omega$ be an $O_0$-connection on $P_{O_0}(\mathbb{H})$. Let $C_W : [0,T] \ni t \longmapsto O(t) \in \mathcal{W}(\mathbb{H})$ be a continuous curve. A lift $C_P$ of $C_W$ is a horizontal lift of $C_W$ with respect to $O_0$, if and only if

$$C_P(t) = s(C_W(t)) \cdot G_s(t),$$

where $s$ is a local section on some $\mathcal{O}$ containing a segment of $C_W$, and $G_s(t) \in \mathcal{G}_{O_0}$ is the solution of

$$\begin{cases} \dfrac{dG_s(t)}{dt} = -\omega_{O(t)}^{s}\left(\dfrac{dO(t)}{dt}\right) \cdot G_s(t), \\ G_s(0) = I. \end{cases}$$

# Expression for the geometric phases of the observable

Fix $O_0 = \{|n\rangle\langle n| : n \geq 1\} \in \mathcal{W}(\mathbb{H})$. For $0 \leq t \leq T$, define

$$\tilde{U}(t) = \sum_{n \geq 1} |\tilde{\psi}_n(t)\rangle\langle n| \in \mathcal{U}(\mathbb{H}),$$

where $|\tilde{\psi}_n(t)\rangle = e^{-\mathrm{i}\int_0^t \langle \psi_n(s), h(s)\psi_n(s)\rangle ds}|\psi_n(t)\rangle$. Then

$$\tilde{C}_P : \ [0, T] \ni t \longmapsto \tilde{U}(t) \in P_{O_0}$$

is the horizontal lift of $C_W : t \to \{|\psi_n(t)\rangle\langle\psi_n(t)| : n \geq 1\}$ associated with $P_{O_0}$ and the canonical connection $\check{\Omega}$, such that

$$\tilde{U}(T) = \sum_{n \geq 1} e^{\mathrm{i}\beta_n} |n\rangle\langle n|$$

is the holonomy element associated with the canonical connection $\Omega$ in $P_{O_0}$.

# Expression for the geometric phases of the observable

**Expression for the geometric phases**

Fix $O_0 = \{|n\rangle\langle n| : n \geq 1\} \in \mathcal{W}(\mathbb{H})$. For any closed lift $\bar{C}_P : [0,T] \ni t \longmapsto \bar{U}(t) \in P_{O_0}$ of $C_W$ associated with $O_0$, i.e., $\bar{U}(T) = \bar{U}(0)$, we have

$$\beta_n = \langle n|\mathrm{i}\int_0^T \bar{U}^\dagger(t)\frac{d}{dt}\bar{U}(t)dt|n\rangle = \langle n|\mathrm{i}\oint_{\bar{C}_P} \bar{U}^\dagger \star d\bar{U}|n\rangle$$

for every $n \geq 1$.

$\beta_n$'s are the geometric phases associated with $C_W$, and independent of the choice of the point $O_0$.

# Remarks

1. Application: Geometric quantum computing, Integer quantum Hall effect.
2. 量子场的 Wightman 公理体系, Feynman 积分的数学理论.
3. 量子电动力学的数学基础.

**Thank you for your attention！**